Possibilistic vs Evidential Valuation Algebra Networks

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***Abstract*—Realistic reasoning applications typically involve many interrelated variables and require the interpretation of data that is both heterogeneous in nature and affected by various types of uncertainty. Accordingly, in this paper we investigate the performance of valuation based algebra networks for reasoning in uncertain multivariate systems. Specifically, we consider networks built from two different approaches to modelling uncertainty: possibility theory and Dempster-Shafer evidence theory. To compare these differing networks, we propose a new possibilistic counterpart to the uncertain implication rule that exists in evidential networks. Using the Captain's decision problem, we analyse the performance of these networks when estimating the number of days a ship will be delayed based on a mixture of uncertain knowledge. We demonstrate that the evidential network is more cautious to changes in uncertainty whereas the possibilistic network is more sensitive. This characteristic could allow the possibilistic network to be used to perform sensitivity analysis on a system.**

***Index Terms*—Computational intelligence, Uncertain multivariate systems, Valuation Based Algebra, Possibility Theory, Evidence Theory, Dempster-Shafer**

I. Introduction

We live in the era of data explosion, where increasingly we rely on machine intelligence for reasoning and decision making. However, such reasoning and decision making is difficult: The data available for reasoning can appear in many forms, such as sensory measurements, prior domain knowledge, human originated observations (spoken or written), or contextual information. Moreover, in most cases, the available data is affected by uncertainty caused by various sources: randomness, incompleteness, ambiguity, or fuzziness [1]. Furthermore, realistic reasoning applications typically involve an interplay of many variables, connected in a network which codifies the relationship between them. Examples of requiring such reasoning over multivariate systems include threat assessment in defence [2], system reliability evaluation [3] and cyber-security [4]. In this paper, we focus on a particular framework for performing reasoning in uncertain multivariate systems known as *valuation algebra*.

First proposed in [5], valuation algebra is a framework for representing knowledge and inferring outcomes within a system. The primary elements of the framework are the *variables* within the system, which we wish to make inferences on, and a set of functions known as *valuations* that represent some knowledge about the relationship between variables. Given these primary elements, inferences are made by manipulating the set of valuations using two operations: *marginalization* and *combination*. Informally, combination represents the aggregation of knowledge and marginalization corresponds to the focusing of knowledge [6]. The process of reasoning is thus the process of marginalizing the combined valuation for each variable in the system [7]. For reference, valuation algebra is a particular form of information algebra introduced in [8]; for a survey see [9].

In order to develop algorithms or networks for reasoning using valuation algebra we need to quantify the uncertainty in the system, i.e. specify the valuation functions. Many uncertainty calculi have been developed for mathematical modelling and processing of uncertain information, such as the probability theory, the theory of possibility [10], imprecise probability theory [11], random set theory [12], Dempster-Shafer (DS) evidence theory [13], [14], etc. Importantly, Shenoy and Shafer [15] established a set of axioms that permit marginalization and combination to be performed via local computations, thus allowing reasoning over multivariate systems to performed in a local, tractable, manner. Among the uncertainty calculi listed, Shenoy and Shafer [15] showed that probability theory, possibility theory and DS evidence theory satisfy the required axioms. Note that reasoning valuation algebra networks that satisfy these axioms are referred to as valuation-based systems (VBS). As a consequence, it is possible to construct differing reasoning networks using different uncertainty calculi.

In this paper we compare two types of valuation based algebra networks, which differ in the way uncertainty is represented and processed. The first network is based on using possibility theory, which we term the possibilistic network, and the second uses DS evidence theory, hence an evidential network. The framework for a VBS evidential network has previously been proposed in [2], [16] however only a partial possibilistic network exists [7], [17], [18]. Specifically, there is no possibilistic counterpart to the evidential uncertain implication rule; an example of such a rule is "if A then B" with a confidence interval. Accordingly, we present a

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new possibilistic transformation for evaluating implications rules allowing direct comparison with the evidential network. Given these networks, we compare the performance using the Captain's decision problem [19]. This problem involves estimating the number of day a ship will be delayed based on a mixture of uncertain knowledge. Using this problem we probe how the networks react to changes in the amount of uncertainty. Our simulations shows that the evidential network is generally more cautious than the possibilistic network. In particular, we show that the possibilistic network is more sensitivity to changes in the implication rule.

The paper is organized as follows. In Section II we outline the Captain's decision problem to motivate reasoning in uncertain multivariate systems. We then in Section III detail valuation based algebra and introduce the framework for an evidential network in Section IV. Next, in Section V, we introduce the framework for a possibilistic network, in particular presenting our possibilistic implication rule. Finally, we present the results of the networks on the Captain's problem in Section VI and end with conclusions.

II. Captain's Decision Problem

To motivate the use of VBA networks, we now introduce the Captain's decision problem [19]. This problem entails a Captain estimating the number of days their ship will be delayed based on uncertain and incomplete knowledge. The knowledge available for reasoning is split into two groups: *Prior* information and *Additional* information. The prior information available to the Captain is as follows:

1) Arrival delay (A) is due to departure delay (D) and the travel delay (T).
2) In 90% of the cases, the departure delay (D) is caused by unexpected difficulties in loading (L) the cargo, or by engine service (S).
3) In 90% of the cases, the travel delay (T) is due to bad weather (W) or unplanned repairs (R) on the sea.
4) A repair on sea (R) is required in 50% to 70% of the trips, if the service (S) has not been carried out prior to the travel.

Next, the additional information, received some time before the departure is:

5) The chance of a one-day loading delay is 30% to 50%.
6) The captain of the ship has decided to skip the engine service on this occasion.
7) According to the weather forecast, there is 60% to 90% chance of bad weather on sea.

In the proceeding we will introduce a possibilitic VBA network for solving this problem and analyse the results.

III. Valuation based Algebra

In this section we will now flesh out the mathematical details behind the elements and operations required in valuation algebra.

The set of all variables in the system for reasoning is defined as the set $\mathcal{V}$. The set of possible values of a set of variables $\mathcal{D} \subseteq \mathcal{V}$ is denoted $\Theta_{\mathcal{D}} = \times\{\Theta_x : x \in \mathcal{D}\}$, where $\times$ represents the Cartesian product, and is referred to as the configurations of $\mathcal{D}$. Given these variables, the function φ is a valuation that represents some knowledge about the relationship among a set of variables $\mathcal{D}$, where $\mathcal{D} \subseteq \mathcal{V}$. The domain of this valuation is the set of variables $\mathcal{D}$ and is obtained using a labeling operation $d : \varphi \to 2^V$, e.g. $d(\varphi) = \mathcal{D}$. The set of all valuations φ whose domain is $d(\varphi) = \mathcal{D}$ is denoted $\Phi_{\mathcal{D}}$. By extension, the set of all valuations with $d(\varphi) \subseteq \mathcal{V}$ is denoted $\Phi = \cup\{\Phi_{\mathcal{D}} : \mathcal{D} \subseteq \mathcal{V}\}$. These concepts can be represented as a graph as illustrated in Fig. 1. The circles represent the variables within the system and the diamonds represent the valuations that link the variables. The domain of a valuation is thus the connected variables. Note that the decision variable, i.e. the variable we wish to infer on, is highlighted in red.

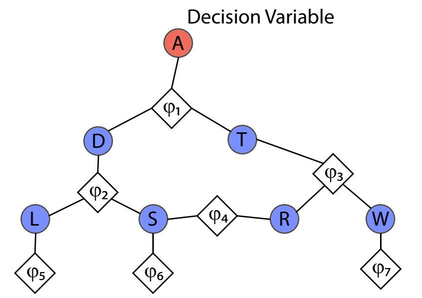


Fig. 1: A graphical representation of the valuation algebra for the Captain's Decision problem described in Section II and analysed in VI.

The problem of inference is first to combine all of the valuations and then marginalize this joint valuation to a subset of the variables that we are interested in. To perform such a task, we need to define the operations combination and marginalization that can be performed on the valuations:

1) **Combination** - is a binary operation defined as $\oplus : \Phi \times \Phi \to \Phi$ such that, if $\varphi_1, \varphi_2 \in \Phi$ are two valuations, then the combined valuation $\varphi_1 \oplus \varphi_2$ represents the aggregated knowledge from φ_1 and φ_2.
2) **Marginalization** - is a binary operation $\downarrow: \Phi \times 2^V \to \Phi$ such that, if $\varphi \in \Phi$ is a valuation and $\mathcal{D} = d(\varphi) \setminus \mathcal{U}$, where $\mathcal{U} \subseteq \mathcal{V}$ and $\setminus$ denotes the set minus operation, then the marginalized valuation $\varphi^{\downarrow \mathcal{D}}$ represents the knowledge about the variables $\mathcal{D}$ implied by φ if we disregard the variables in $\mathcal{U}$ [7].

Using these operations, a straightforward approach to inference is to first compute the joint valuation $\oplus\Phi = \varphi_1 \oplus \cdots \oplus \varphi_r$, assuming a finite set of valuations $\Phi = \{\varphi_1, \ldots, \varphi_r\}$, and then to marginalize it to the domain of interest $\mathcal{D}^o$. The problem with this approach however is that the number of variables increases with each combination. For example, if there are n variables in $\mathcal{V}$ and the cardinality of the configuration space for each variable is m, then the configuration of joint domain of all variables, $\Theta_{\mathcal{V}}$, has a cardinality of m^n, which soon becomes intractable to calculate.

The solution to this problem is to compute the marginal $(\oplus\Phi)^{\downarrow \mathcal{D}^o}$ on local domains and obtain the same result without having to compute $\oplus\Phi$ explicitly. This local approach is acceptable if, and only if, the labeling, combination and marginalization operations satisfy a set of axioms [8], [15], [20], [21]. A detailed list and explanation of the axioms is given [8], [22]. If the operations satisfy the required axioms then the system $\{V, \Phi, d, \oplus, \downarrow\}$ is referred to as a valuation algebra system, i.e. a valuation-based system (VBS).

Finally, the process by which inference is performed in a VBS is known as a Fusion algorithm [20], [21]; for a detailed explanation we refer the reader to [21], [22]. In brief, fusion algorithms work by successively eliminating all the variables $X \in \mathcal{N}$, where $\mathcal{N} = \mathcal{V} \setminus \mathcal{D}^o$ is the set of variables which are of no interest, such that one is left with the marginal relating to $\mathcal{D}^o$. Interestingly, due to axioms of valuation algebra, the order in which the variables are eliminated does not affect the final result. However, different elimination sequences can have different computational costs. Finding an optimal elimination sequence is an NP-complete problem [20], but there exist several heuristics methods [19], [23], [24].

IV. Evidential Networks

In the previous discussion we described VBS in the abstract – a system $\{V, \Phi, d, \oplus, \downarrow\}$ that satisfies certain axioms. We now move to the concrete and introduce *evidential networks* [2], [22]. An evidential network is a VBS built using Dempster-Shafer's theory of evidence [13], [14]; the valuations are expressed using belief functions and the operations used to manipulate them adhere to the transferable belief model (TBM) [25]. Note that the theory of evidence satisfies the axioms of a valuation algebra [8]. In the following we review the main components and tools required in an evidential network described in [2], [16], [22]

We start by introducing belief functions. Let $\Theta_{\mathcal{D}}$ denote the finite set of configurations for the variables in $\mathcal{D}$ in an evidential network. Note that the elements of $\Theta_{\mathcal{D}}$ are assumed to be mutually exclusive and exhaustive. The beliefs about the true values of $\mathcal{D}$ are expressed on subsets A of $\Theta_{\mathcal{D}}$ using basic belief assignments (BBA). The BBA $m^{\mathcal{D}}$ is a multivariate function on the domain $\mathcal{D}$ which assigns every subset A of $\Theta_{\mathcal{D}}$ a value in $[0, 1]$ such that $m^{\mathcal{D}} : 2^{\Theta_{\mathcal{D}}} \rightarrow [0, 1]$. Note that $2^{\Theta_{\mathcal{D}}}$ is the power set of $\Theta_{\mathcal{D}}$. Importantly, BBA's satisfy the following condition: $\sum_{A \subseteq \Theta_{\mathcal{D}}} m^{\mathcal{D}}(A) = 1$. The subsets A that have a belief $m^{\mathcal{D}}(A) > 0$ are referred to as focal sets of the BBA. Finally, the state of complete ignorance is denoted as the *vacuous* BBA: $m^{\mathcal{D}}(A) = 1$ if $A = \Theta_{\mathcal{D}}$.

Evidential Combination:
Using BBA's, the combination operator is achieved using Dempster's rule of combination [13]. Formally, let φ_1 and φ_2 represent two valuations with the same domain $d(\varphi_1) = d(\varphi_2) = \mathcal{D}$. The corresponding BBA's $m_1^{\mathcal{D}}$ and $m_2^{\mathcal{D}}$ can be combined using the following equation:

$$\left(m_1^{\mathcal{D}} \oplus m_2^{\mathcal{D}}\right)(A) = \frac{\sum\limits_{B \cap C = A} m_1^{\mathcal{D}}(B) \cdot m_2^{\mathcal{D}}(C)}{1 - \sum\limits_{B \cap C = \varnothing} m_1^{\mathcal{D}}(B) \cdot m_2^{\mathcal{D}}(C)} \quad (1)$$

where $A, B, C \subseteq \Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{D}}$ is set of all configurations of $\mathcal{D}$.

Evidential Marginalization:
Marginalization is the projection of a BBA onto a coarser domain. Consider a valuation φ with a domain $d(\varphi) = \mathcal{D}$ and a belief function $m^{\mathcal{D}}$. The marginalization of $m^{\mathcal{D}}$ onto a coarser domain $\mathcal{D}' \subseteq \mathcal{D}$ is defined as

$$m^{\downarrow \mathcal{D}'}(A) = \sum_{B \downarrow A} m^{\mathcal{D}}(B) \quad (2)$$

where the summation above is over all $B \subseteq \Theta_{\mathcal{D}}$ such that the configurations in B reduce to the configuration in $A \subseteq \Theta_{\mathcal{D}'}$ by the elimination of variables $\mathcal{D} \setminus \mathcal{D}'$.

Along with the above equations we require two additional operations in order to perform inference in an evidential VBS network. The first operation is known as *vacuous extension* and is required if we need to combine two valuations φ_1 and φ_2 with different domains $d(\varphi_1) = \mathcal{D}_1$ and $d(\varphi_2) = \mathcal{D}_2$. The second operation is to convert uncertain implication rules into valuations with a known belief function. Such rules allow the expression of expert knowledge in the form "if A then B" with a certain degree of confidence [2]. Note that, in relation to the three cases of implication operations identified in [26], our approach to this rule falls into the "classical view of implication".

Evidential Vacuous Extension:
Consider a valuation φ on a domain $d(\varphi) = \mathcal{D}_1$ with a BBA $m^{\mathcal{D}_1}$. To combine this BBA with another defined on a domain $\mathcal{D}_2$ such that $\mathcal{D}_1 \neq \mathcal{D}_2$, we need the following operation to extend the BBA onto the joint domain $\mathcal{D}_1 \cup \mathcal{D}_2$:

$$m^{\mathcal{D}_1 \uparrow (\mathcal{D}_1 \cup \mathcal{D}_2)}(C) = \begin{cases} m^{\mathcal{D}_1}(A) & \text{if } C = A \times \Theta_{\mathcal{D}_2}, A \subseteq \Theta_{\mathcal{D}_1} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where $\Theta_{\mathcal{D}_1}$ and $\Theta_{\mathcal{D}_2}$ denote respectively the configuration sets corresponding to the domains $\mathcal{D}_1$ and $\mathcal{D}_2$.
Remark: Marginalization is the inverse operation of vacuous extension however, in general, vacuous extension is not the inverse of marginalization.

Evidential Uncertain Implication Rule:
Finally, we review the evidential uncertain implication rule proposed in [16]. The authors exploited the logical equivalence of "A implying B" and "not A or B" to obtain a belief function. Formally, consider two domains $\mathcal{D}_1$ and $\mathcal{D}_2$ related by an implication rule R of the form:

$$A \subseteq \Theta_{\mathcal{D}_1} \;\Rightarrow\; B \subseteq \Theta_{\mathcal{D}_2}, \quad (4)$$

where $\Theta_{\mathcal{D}_1}$ and $\Theta_{\mathcal{D}_2}$ are the corresponding configurations. Let $[\alpha, \beta]$ represent the confidence interval for this rule where $0 \leq \alpha < \beta \leq 1$. The valuation representing this rule is expressed on the joint space $\mathcal{D}_1 \cup \mathcal{D}_2$ with the following BBA:

$$m_R^{\mathcal{D}_1 \times \mathcal{D}_2}(C) = \begin{cases} \alpha & \text{if } C = (A \times B) \cup (A^c \times \Theta_{\mathcal{D}_2}) \\ 1 - \beta & \text{if } C = (A \times B^c) \cup (A^c \times \Theta_{\mathcal{D}_2}) \\ \beta - \alpha & \text{if } C = \Theta_{\mathcal{D}_1 \cup \mathcal{D}_2} \end{cases} \quad (5)$$

where A^c is the complement of A in $\Theta_{\mathcal{D}_1}$ and B^c is the complement of B in $\Theta_{\mathcal{D}_2}$.

We conclude this section by noting that in order to make decisions using evidential theory the belief functions needs to be mapped to probabilities. Several transformations exist however in this paper we shall focus on the pignistic [27] and Voorbraak [28] transformations.

V. Possibilistic Networks

In this section we now introduce a *possibilistic network* for VBS whereby valuations are represented using possibility functions and the combination and marginalization operations adhere to possibility theory. An initial possibilistic VBS network was proposed in [7], [17]. In particular, Shenoy [7] proved that the possibilistic versions of combination and marginalization satisfy the set of axioms required for local computation of the marginal $(\oplus\Phi)^{\downarrow\mathcal{D}^o}$ without having to explicitly compute $\oplus\Phi$. Accordingly, the following definitions of possibilistic combination and marginalization form a valid VBS and can be solved using a fusion algorithm. To accompany these operations, we extend the possibilistic network by proposing a possibilistic version of the uncertain implication rule capable of dealing with interval uncertainties (or confidence intervals). We also include possibilistic vacuous extension. In the following we define these possibilistic operations.

Similar to the previous section, we review the basics of possibility theory introduced by Zadeh [10] and later expanded upon in [29], [30]. Let $\Theta_{\mathcal{D}}$ denote the configuration set for the variables in $\mathcal{D}$ – again each element in $\Theta_{\mathcal{D}}$ is assumed mutually exclusive and exhaustive. The possibility measure of an event $A \subseteq \Theta_{\mathcal{D}}$ is a mapping $\Pi : 2^{\Theta_{\mathcal{D}}} \rightarrow [0,1]$, where $2^{\Theta_{\mathcal{D}}}$ is the power set of $\Theta_{\mathcal{D}}$. This mapping Π satisfies the following axioms: 1) $\Pi(\varnothing) = 0$ – the set $\Theta_{\mathcal{D}}$ is an exhaustive set of configurations; 2) $\Pi(\Theta_{\mathcal{D}}) = 1$ – the mapping Π is free of contradictions; 3) $\Pi(A_1 \cup A_2) = \max(\Pi(A_1), \Pi(A_2))$ – this replaces the additivity axiom in probability theory.

Having introduced a measure of possibility, we can now introduce a possibility function $\pi : \Theta_{\mathcal{D}} \rightarrow [0,1]$ such that

$$\Pi(A) = \max_{x \in A} \pi(x) \tag{6}$$

for every $A \subseteq \Theta_{\mathcal{D}}$. This function can be used to represent a range of knowledge starting with complete knowledge, where $\pi(x) = 1$ for $x = x_0$ and 0 otherwise, to complete ignorance, where $\pi(x) = 1, \ \forall x \in \Theta_{\mathcal{D}}$. A possibility can be converted to a probability and vice versa using the following transformation:

$$\pi(x) = \frac{p(x)}{\max_{x \in \Theta_{\mathcal{D}}} p(x)} \Longleftrightarrow p(x) = \frac{\pi(x)}{\sum_{x \in \Theta_{\mathcal{D}}} \pi(x)}. \tag{7}$$

Importantly, in contrast with belief functions, possibility functions operate on singletons only. Thus we can expect a difference in behaviour when comparing evidential networks to possibilistic ones.

Finally, to finish the review, we introduce the concept of *necessity* – the dual of possibility. The duality of possibility and necessity can be expressed by $N(A) = 1 - \Pi(A^c)$ where A^c is the complement of A in $\Theta_{\mathcal{D}}$. Importantly, a possibility function π induces both Π, see (6) and N, as follows: $N(A) = \min_{x \in A^c}(1 - \pi(x))$. Accordingly, the necessity/possibility pair can be interpreted as a *lower* and *upper* confidence in the sense of Walley's upper and lower previsions [11]. We shall use this duality when dealing with knowledge represented using a confidence interval.

Possibilistic Combination:
Consider two valuations φ_1 and φ_2 with the same domain $d(\varphi_1) = d(\varphi_2) = \mathcal{D}$. Let $\pi_1^{\mathcal{D}}$ and $\pi_2^{\mathcal{D}}$ be the possibility functions representing valuations φ_1 and φ_2, respectively and let $\Theta_{\mathcal{D}}$ represent the set of all configurations corresponding to the domain $\mathcal{D}$. The combination of the two possibility functions is thus defined as:

$$\left(\pi_1^{\mathcal{D}} \oplus \pi_2^{\mathcal{D}}\right)(a) = \begin{cases} K^{-1} \pi_1^{\mathcal{D}}(a) \cdot \pi_2^{\mathcal{D}}(a) & \text{if } K \neq 0 \\ 0 & \text{if } K = 0 \end{cases}, \tag{8}$$

$\forall a \in \Theta_{\mathcal{D}}$, where K is a normalisation constant defined as

$$K = \max_{a \in \Theta_{\mathcal{D}}} \left\{ \pi_1^{\mathcal{D}}(a) \cdot \pi_2^{\mathcal{D}}(a) \right\}.$$

Note that this constant is required so that the result of the combination is a valid possibility function.
Remark: In Zadeh's [10] possibility theory, the combination[1] of two possibility functions can be achieved using either minimization or multiplication. However, as pointed out in [20], in both cases normalization is required to satisfy the VBS axioms and minimization followed by normalization is not associative, which is a requirement of another VBS axiom.

Possibilistic Marginalization:
Consider a valuation φ with a domain $d(\varphi) = \mathcal{D}$ and a possibility function $\pi^{\mathcal{D}}$. Let $\mathcal{D}'$ represent a coarser domain such that $\mathcal{D}' \subseteq \mathcal{D}$ and let $\Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{D}'}$ represent respectively the set of all configurations corresponding to the domains $\mathcal{D}$ and $\mathcal{D}'$. The marginalization of the possibility function onto the domain $\mathcal{D}'$ is defined as:

$$\pi^{\downarrow \mathcal{D}'}(a) = \max_{b \in \Theta_{\mathcal{D}}} \left\{ \pi^{\mathcal{D}}(a, b) \right\}, \quad \forall \quad a \in \Theta_{\mathcal{D}'}. \tag{9}$$

Remark: In Zadeh's [10] possibility theory marginalization is referred to as projection.

Possibilistic Vacuous Extension:
Consider a valuation φ with a domain $d(\varphi) = \mathcal{D}_1$ and a possibility function $\pi^{\mathcal{D}_1}$. Let $\mathcal{D}_2$ represent a different domain such that $\mathcal{D}_1 \neq \mathcal{D}_2$ and let $\Theta_{\mathcal{D}_1}$ and $\Theta_{\mathcal{D}_2}$ represent respectively the set of all configurations corresponding to the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. The vacuous extension of the possibility function $\pi^{\mathcal{D}_1}$ onto the domain $\mathcal{D}_1 \cup \mathcal{D}_2$ is defined as:

$$\pi^{\mathcal{D}_1 \uparrow (\mathcal{D}_1 \times \mathcal{D}_2)}(c) = \begin{cases} \pi^{\mathcal{D}_1}(a) & \text{if } c = a \times \Theta_{\mathcal{D}_2}, \ a \in \Theta_{\mathcal{D}_1} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

[1] Combination is termed particularization by Zadeh [10].

TABLE I: List of variables in the Captain's Decision Problem.

Variable Name	Notation	Possible Configurations
Loading delay	L	$\Theta_L = \{0, 1\}$
Service	S	$\Theta_S = \{0, 1\}$
Weather	W	$\Theta_W = \{0, 1\}$
Repair	R	$\Theta_R = \{0, 1\}$
Departure delay	D	$\Theta_D = \{0, 1, 2\}$
Travel delay	T	$\Theta_T = \{0, 1, 2\}$
Arrival delay	A	$\Theta_A = \{0, 1, 2, 3, 4\}$

Possibilistic Uncertain Implication Rule:
We now introduce our possibilistic uncertain implication rule. The rule is devised by interpreting the associated confidence interval as a necessity-possibility pair. Formally, consider two domains $\mathcal{D}_1$ and $\mathcal{D}_2$ related by an implication rule R of the form:

$$a \in \Theta_{\mathcal{D}_1} \;\Rightarrow\; b \in \Theta_{\mathcal{D}_2}. \tag{11}$$

Note that as we are dealing with possibilities we have singletons. Again, let us associate the following confidence interval $[\alpha, \beta]$ for this rule where $0 \leq \alpha < \beta \leq 1$. The valuation representing this rule is expressed using the following possibility function:

$$\pi_R^{\mathcal{D}_1 \times \mathcal{D}_2}(c) = \begin{cases} \beta K^{-1} & \text{if } c = a \times b \\ (1-\alpha)K^{-1} & \text{if } c = a \times b^c \\ 1 & \text{if } c = a^c \times \Theta_{\mathcal{D}_2} \end{cases} \tag{12}$$

where $K = \max(\beta, 1-\alpha)$ and $\Theta_{\mathcal{D}_2}$ is the configurations for $\mathcal{D}_2$. Note that the possibility for $a \times b^c$ is obtained from the necessity $N(a, b^c) = \alpha$.

Remark: Interestingly, in the possibility framework, the calculation of either the marginal $\left(\pi_R^{\mathcal{D}_1 \times \mathcal{D}_2}\right)^{\downarrow \mathcal{D}_1}$ or $\left(\pi_R^{\mathcal{D}_1 \times \mathcal{D}_2}\right)^{\downarrow \mathcal{D}_2}$ result in complete ignorance suggesting that, on its own, the implication rule is uninformative.

VI. Analysis of the Captain's Decision Problem

In this section, we now analyse and compare the performance of a possibilistic network to that of an evidential network when solving the Captain's Decision problem stated in Section II. For completeness, we include the decisions obtained from the evidential network when using either the pignistic or Voorbraak transformation.

A. Posing the Problem

The full set of variables for the Captain's problem is $\mathcal{V} = \{L, S, W, R, D, T, A\}$ and each piece of prior and additional knowledge represents a valuation such that $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_7\}$. For reference, we shall refer to the set of valuations corresponding to the prior knowledge as $\Phi_{\text{prior}} = \{\varphi_1, \ldots, \varphi_4\}$ and the valuations relating to additional knowledge $\Phi_{\text{add}} = \{\varphi_5, \ldots, \varphi_7\}$. The variables and their associated possible configurations (expressed as the number of days of delay) are listed in Table I. A graphical representation of the valuation algebra for this example is shown in Fig. 1.

The next step is to convert the knowledge listed in the problem into valuations represented by functions. Note that, for brevity, we detail this process for the possibilistic network only. Let us start by defining the corresponding possibility functions in Φ_{prior}. The first possibility function, π_1, is expressed over the space $\Theta_{D \times T \times A} = \{(d, t, a) : d \in \Theta_D; t \in \Theta_T; a \in \Theta_A\}$ and defined as:

$$\pi_1(d, t, a) = \begin{cases} 1 & \text{if } d + t = a \\ 0 & \text{otherwise.} \end{cases}$$

The second possibility function, π_2, is expressed over the space $\Theta_{L \times S \times D} = \{(l, s, d) : l \in \Theta_L; s \in \Theta_S; d \in \Theta_D\}$ and defined as:

$$\pi_2(l, s, d) = \begin{cases} 1 & \text{if } l + s = d \\ 1/9 & \text{otherwise.} \end{cases}$$

The third possibility function, π_3, is expressed over the space $\Theta_{W \times R \times T} = \{(w, r, t) : w \in \Theta_W; r \in \Theta_R; t \in \Theta_T\}$ and takes the same form as π_2, that is:

$$\pi_3(w, r, t) = \begin{cases} 1 & \text{if } w + r = t \\ 1/9 & \text{otherwise.} \end{cases}$$

The fourth possibility function, π_4, is expressed over the space $\Theta_{S \times R} = \{(s, r) : s \in \Theta_S; r \in \Theta_R\}$ and is obtained using the uncertain implication rule (12). The rule states that with confidence $[0.5, 0.7]$ $s = 0 \Rightarrow r = 1$. Thus, $\alpha = 0.5$ and $\beta = 0.7$, resulting in

$$\pi_4(s, r) = \begin{cases} 0.5/0.7 & \text{if } (s, r) = (0, 0) \\ 1 & \text{otherwise.} \end{cases}$$

Considering the additional knowledge valuations Φ_{add}, the possibility function π_5 is expressed over Θ_L and is defined as

$$\pi_5(l) = \begin{cases} 1 & \text{if } l = 0 \\ 5/7 & \text{if } l = 1. \end{cases}$$

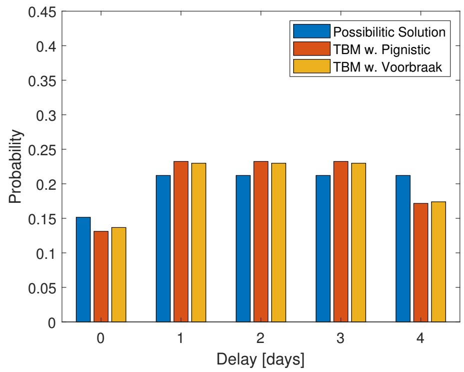
This function is obtained by converting an interval probability to a possibility. The sixth possibility function π_6 is expressed over Θ_S and corresponds to an exact, certain, statement. Accordingly, the function represents complete knowledge:

$$\pi_6(s) = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s = 1. \end{cases}$$

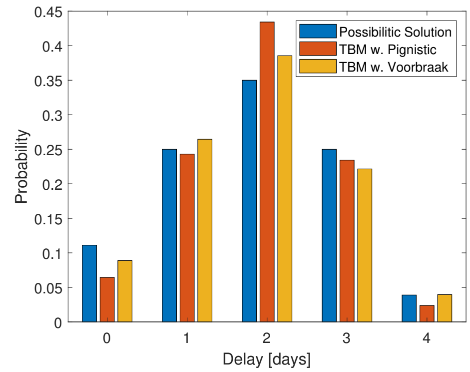
Finally, the seventh possibility function π_7 is expressed over Θ_W and again the corresponding piece of information involves an interval probability thus we have:

$$\pi_7(w) = \begin{cases} 4/9 & \text{if } w = 0 \\ 1 & \text{if } w = 1. \end{cases}$$

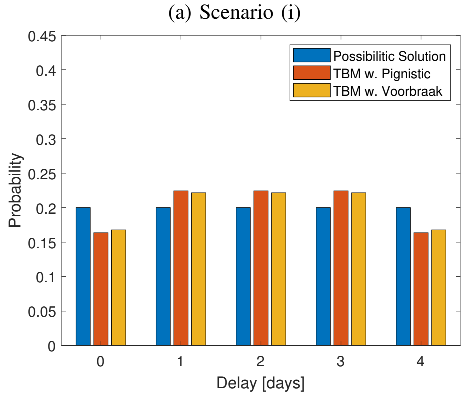
Given the possibility functions defined above and the equivalent functions for the evidential networks, we can now use a fusion algorithm to obtain the posterior probabilities for each element in Θ_A.



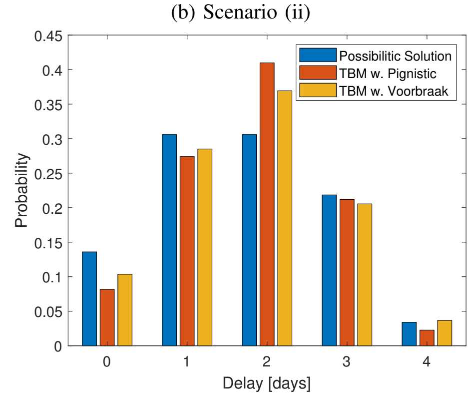
(a) Scenario (i)



(b) Scenario (ii)



(c) Scenario (iii)



(d) Scenario (iv)

Fig. 2: Comparison of the probabilities associated with arrival delay for the Captain's decision problem computed using three different VBS networks: a possibilistic network, an evidential network with the pignistic transform (TBM w Pignistic), and an evidential network with the Voorbraak transform (TBM w Voorbraak) . Graphs (a) and (b) correspond to original problem description in Section II and graphs (c) and (d) correspond to the same problem but with probability interval on statement 4 changed to 30% to 70%. Graphs (a) and (c) shows the results when only prior knowledge is used to compute the probabilities and graphs (b) and (d) shows the results when prior and additional knowledge is used.

B. Simulation Results

We start by considering the performance of the networks on two scenarios: (i) using only the prior knowledge Φ_{prior} and assuming complete ignorance for the valuations in Φ_{add}; (ii) using both prior knowledge Φ_{prior} and the additional knowledge Φ_{add}. The resulting probabilities for each scenario, using each type of network, are shown in Fig. 2; the graph in Fig. 2(a) corresponds to scenario (i) and the graph in Fig. 2(b) corresponds to scenario (ii).

The resulting probabilities illustrated in Fig. 2 show first that the prior knowledge represented in Φ_{prior} on its own is fairly uninformative. In contrast, the posterior probabilities relating to the arrival delay dramatically change with the addition of Φ_{add}. This observation holds for all of the networks tested.

If we focus on the results for the possibilistic network. For scenario (i), the probabilities associated with delays greater than 1 day are all equal and higher than no delay whereas, in scenario (ii), a delay of 2 days is most likely with $p_{\text{poss}}(2) = 0.350$. A similar pattern is observed when using the evidential network. For scenario (i) a delay of 0 days is least likely followed by a delay of 4 days then the rest are equally likely. Likewise, for scenario (ii), the evidential networks find a delay of 2 days is most likely; the probability using the pignistic transform is $p_{\text{pig}}(2) = 0.434$ whereas the Voorbraak probability is $p_{\text{voor}}(2) = 0.386$. The graphs also show that there is a slight difference between the Voorbraak and pignistic probabilities but they are more similar to themselves than the possibilistic probabilities. Overall, however, decisions based on any of the networks will be similar for these scenarios.

To further probe the performance of the networks, we now reduce the lower confidence in statement 4 of the Captain's Problem – the statement is now "A repair on sea (R) is required in **30%** to 70% of the trips, if the service (S) has not been carried out prior to the travel" – and recalculate the posterior probabilities. For these new simulations we refer to using only Φ_{prior} and assuming complete ignorance for Φ_{add} as scenario (iii) and using the both sets of knowledge as scenario (iv). The resulting probabilities for these new scenarios are also shown in Fig. 2; the graph in Fig. 2(c) corresponds to scenario (iii) and the graph in Fig. 2(d) corresponds to scenario (iv).

The resulting probabilities for these new scenarios again show the benefit of the additional knowledge in decision making. However, the distribution of the probabilities generated by the possibilistic network has changed compared to the evidential networks. In scenario (iii), reducing the lower confidence in the implication rule results in the possibilistic network assigning an equal probability to all of the configurations in Θ_A. Similarly, in scenario (iv), the possibilistic network now predicts that a delay of 1 day is equally as likely as a delay of 2 days; the corresponding probabilities are $p_{\text{poss}}(1) = p_{\text{poss}}(2) = 0.306$. In contrast, although the probabilities predicted by the evidential networks for each

TABLE II: Probability of requiring a repair at sea verses not requiring one in the Captain's problem.

VBS Network	Scenario (ii)		Scenario (iv)	
	Repair	No Repair	Repair	No Repair
Possibilistic	0.583	0.417	0.500	0.500
Evidential with Pignistic	0.750	0.250	0.650	0.350
Evidential with Voorbraak	0.667	0.333	0.588	0.412

scenario have changed, their distribution is relatively similar to the previous scenarios. As a consequence, decisions based on the possibilistic network are now different than those based on either evidential network.

This difference in behaviour stems from the implication rules used in each network. The possibilistic rule assigns values to each individual outcome whereas the evidential rule assigns belief to overlapping sets of outcomes, including complete ignorance. To examine the affect of these differing approaches, let us consider just the valuations that relate to the variables service S and repair R and determine the probability of requiring a repair at sea by marginalizing to R. The resulting probabilities for scenario (ii) and (iv) are shown in Table II. The table shows that the evidential networks are more cautious; they predict a repair is more likely in both scenarios. In contrast, the possibilistic network initially predicts a repair is more likely in scenario (ii) but then changes to predict a repair being as likely as no repair in scenario (iv). The possibilistic network is thus more sensitive to the change in the implication rule than the evidential networks. This sensitivity of the possibilistic network accounts for the different probability distributions in Fig. 2.

VII. Conclusions

In this paper, we have investigated the performance of valuation based algebra networks when making decisions in multivariate systems. Specifically, we introduced VBS networks built from two uncertainty calculi: possibility theory and Dempster-Shafer evidence theory. For the possibilistic network, we presented a new transformation for converting uncertain implication rules into a possibility function, which allowed the network to evaluate statements such as "if a then b" with a certain confidence interval. The performance of these networks were then analysed when estimating the delay arrival (in days) in the Captain's decision problem. In more detail, we simulated four scenarios of the problem with differing amounts of uncertainty. Our simulation results showed that the possibilistic network was more sensitive to changes in the uncertain implication rule than the evidential network. Such a trait could be advantageous when analysing the sensitivity of a system to changes in uncertainty, i.e. testing how robust a system is. In future work we intend to investigate this potential application and develop a general framework for evaluating VBS networks, which is currently an open problem.

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