Finding the Minimum Rate of Innovation in the Presence of Noise

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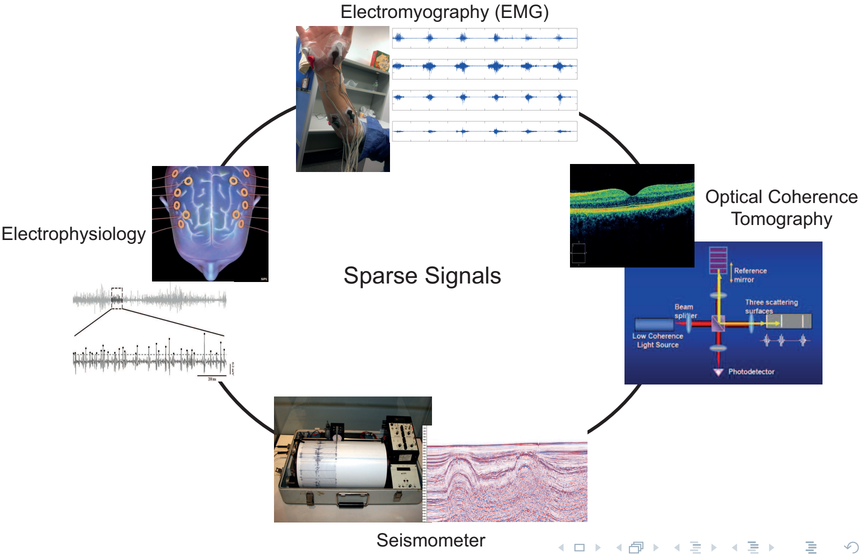
Outline

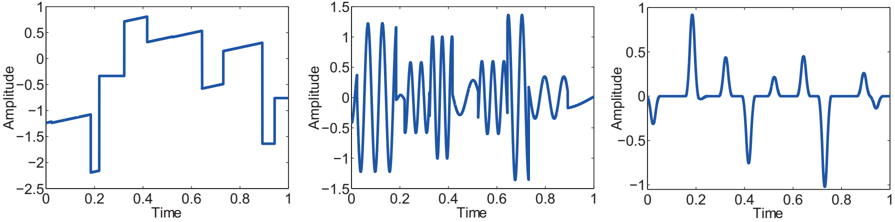
1. Signal Acquisition $\Longrightarrow$ Real World to Digital World
2. Signals with Finite Rate of Innovation (FRI)
   - General Framework
   - Reconstruction Procedure
   - Model Mismatch
3. Determining the Minimum Rate of Innovation
   - Model-Fitting Approach to FRI
   - Fitting based on a MSE Budget
   - Fast Estimation Algorithm
4. Simulation Results
5. Conclusion

Acquiring Sparse Signals



Representing Sparsity with Finite Rate of Innovation

Sparse signals $\iff$ FRI signals



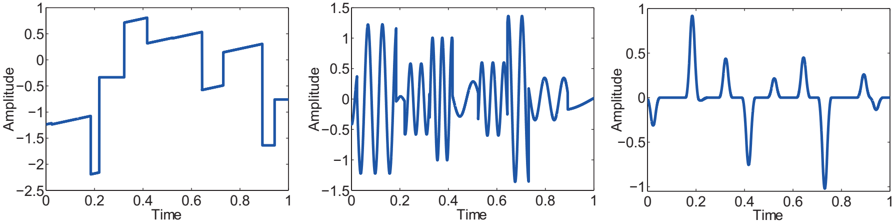
(a) Piecewise Polynomials (b) Piecewise Sinusoids (c) Stream of Pulses

Characteristics:

Finite number of sparse, unpredictable, parameters

Representing Sparsity with Finite Rate of Innovation

Sparse signals $\Longleftrightarrow$ FRI signals



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Characteristics:

$$\underbrace{\text{Finite number of sparse, unpredictable, parameters}}_{\text{Innovations of the Signals}}$$

Representing Sparsity with Finite Rate of Innovation

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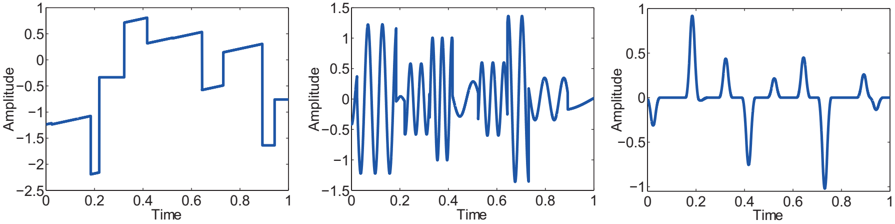

(a) Piecewise Polynomials (b) Piecewise Sinusoids (c) Stream of Pulses

Applications in Bio-medics:

- Spike Estimation in Neurophysiological Data[1]
- Fetal heart rate detection[2]

[1] J. Oñativia, R. Schultz & P.L. Dragotti, '*A finite rate of innovation algorithm for fast and accurate spike detection from two-photon calcium imaging*', J. Neural Engineering, 2013.
[2] A Nair & P. Marziliano, '*Fetal heart rate detection using VPW-FRI*', ICASSP, 2014

Representing Sparsity with Finite Rate of Innovation

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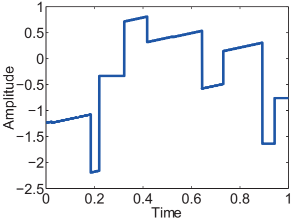

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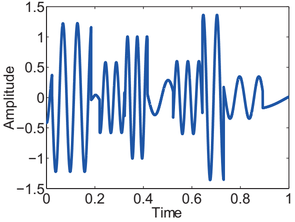
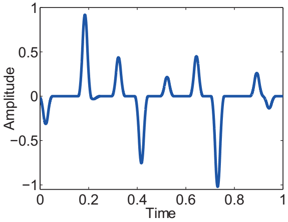


(b) Piecewise Sinusoids



(c) Stream of Pulses

Two important questions...

- How to find the parameters?
- How many parameters to find? $\Longleftrightarrow$ Rate of Innovation

General FRI Framework

All FRI signals can be reduced to[1,2]....

$$\underbrace{\text{Sum of Diracs}}_{\sum_{k=1}^{K} x_k \delta(t - t_k)} \quad \underset{\text{Transform}}{\overset{\text{Linear}}{\Longleftrightarrow}} \quad \underbrace{\text{Sum of Sinusoids}}_{\sum_{k=1}^{K} x_k \mathrm{e}^{-j\omega t_k}}$$

[1] P.L. Dragotti, M. Vetterli & T. Blu '*Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strang-Fix*', IEEE Trans. Signal Processing, 2007.
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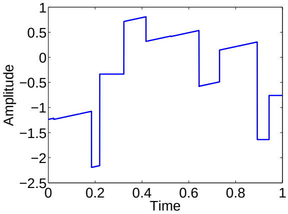

Finite Differences:

Discontinuities $\Longrightarrow$ Diracs

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↬ Estimate FRI parameters: $\begin{cases} x_k & \text{Amplitudes} \\ t_k & \text{Positions} \end{cases}$

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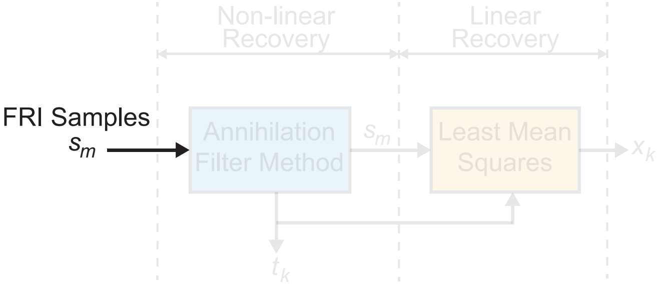
Reconstruction Procedure

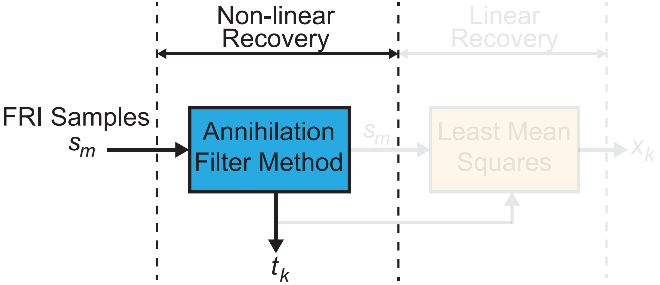


Uniformly sample:

FRI samples: $$\underbrace{s_m = \sum_{k=1}^{K} x_k \mathrm{e}^{-j2\pi m t_k}}_{\text{Sum of Sinusoids}}$$

N samples in total.

Reconstruction Procedure

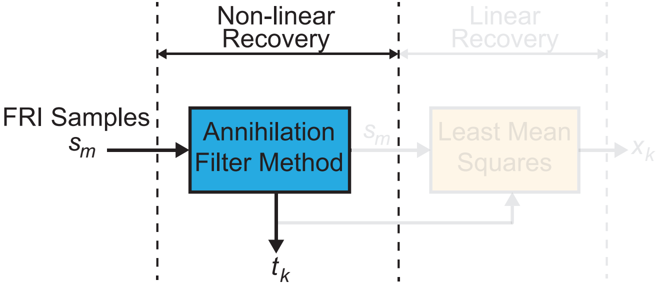


Step 1) Non-linear Recovery of t_k

Determine filter $H(z) = \sum_{k=0}^{K} h_k z^{-k}$ such that $\underbrace{h_m * s_m = 0}_{\text{Annihilation}}$

↬ Roots of filter H define the positions t_k

Reconstruction Procedure

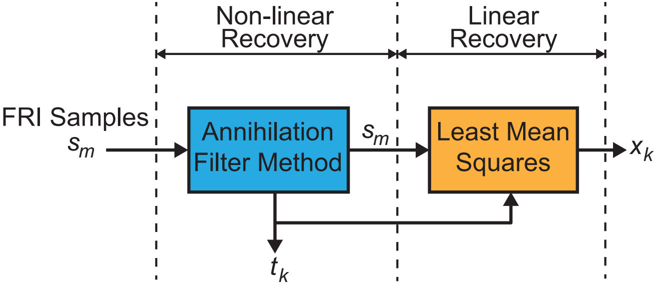


In more detail:

$$h_m * s_m = 0 \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} s_K & s_{K-1} & \cdots & s_0 \\ s_{K+1} & s_K & \cdots & s_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_{N-1} & s_{N-2} & \cdots & s_{N-K-1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{K+1} \end{bmatrix}}_{\mathbf{h}} = 0$$

Solve Annihilation Equation: $\mathbf{A}\mathbf{h} = 0 \longrightarrow$ Requires $N = 2K$ samples

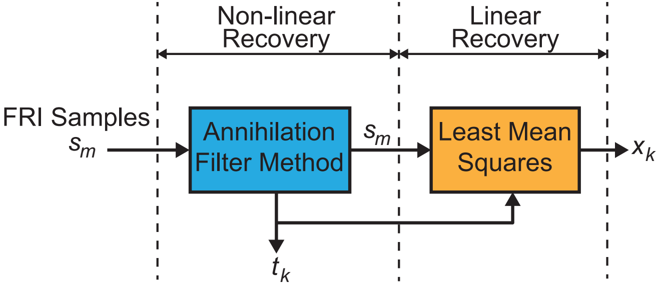
Reconstruction Procedure



Step 2) Linear Recovery of x_k

Using $t_k \implies$ Determine x_k via least mean squares

Reconstruction Procedure



Estimating the rate of innovation:

1. Over-estimate rate $\implies$ Set $K_{\text{est}} = N/2$
2. Build the annihilation matrix $\mathbf{A}$
3. Rank of $\mathbf{A}$ = Actual K

Model Mismatch

An imperfect world:

Sample Corruption: $\tilde{s}_m = s_m + \epsilon_m$ therefore $\tilde{\mathbf{A}}\mathbf{h} \neq 0$

Model Mismatch

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Sample Corruption: $\tilde{s}_m = s_m + \epsilon_m$ therefore $\tilde{\mathbf{A}}\mathbf{h} \neq 0$

Many methods to estimate sinusoids in noise:

For example: Total Least Squares,
Cadzow's Method[1],
Matrix Pencil[2],
⋮

[1] J. Cadzow, '*Signal Enhancement - A composite property mapping algorithm*', IEEE Acoust., Speech and Processing, 1988
[2] Y. Hua & T.K. Sarkar, '*Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise*', IEEE Acoust., Speech and Processing, 1990

Model-Fitting approach

Noiseless Conditions:

Inverse Fourier Transform of the sum of sinusoids s_m

$\Updownarrow$

Expressed as a ratio of two Polynomials, P and H[1]

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In other words:

$$s_m \xleftrightarrow[\text{Transform}]{\text{Inverse Fourier}} y_n = \frac{P\left(e^{j\omega_n}\right)}{H\left(e^{j\omega_n}\right)}$$

where

- $P\left(e^{j\omega_n}\right)$ is a polynomial of order $K-1$
- $H\left(e^{j\omega_n}\right)$ is the annihilation filter, a polynomial of order K
- $\omega_n = 2\pi n/N$

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$\looparrowright$ Samples y_n completely defined by the coefficients of the polynomials

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↬ Departs from the Annihilation Equation!!

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Model Mismatch $\Longrightarrow$ Fit ratio model to the noisy samples $\tilde{y}_n$

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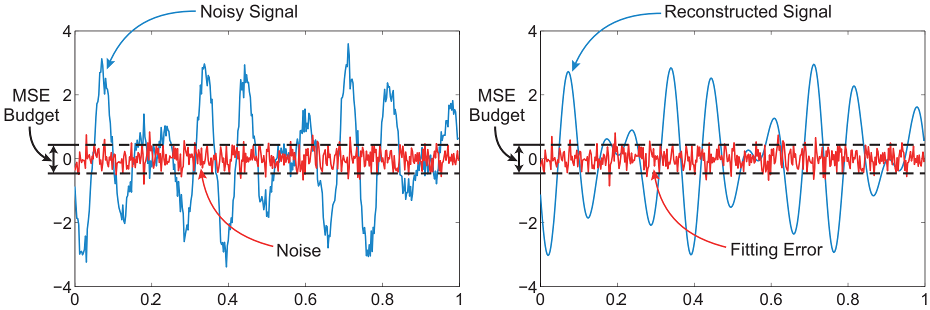
Fitting using a MSE Budget



Central Concept:

Fit model to the noisy data until the MSE budget has been satisfied

$$\looparrowright \quad \|\text{Data} - \text{Model}\|_2^2 \;\leq\; \text{MSE Budget}$$

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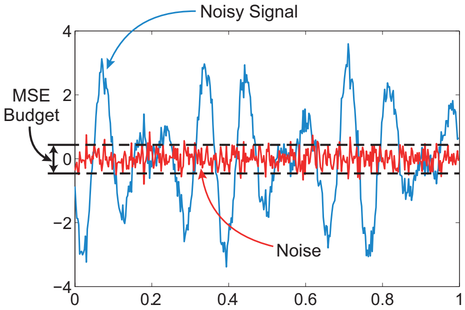

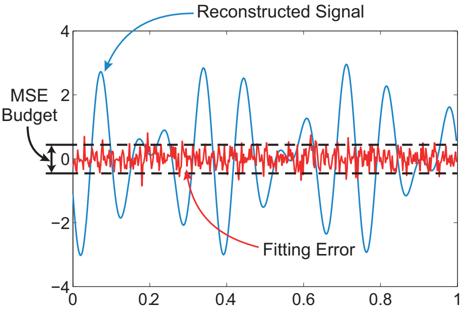

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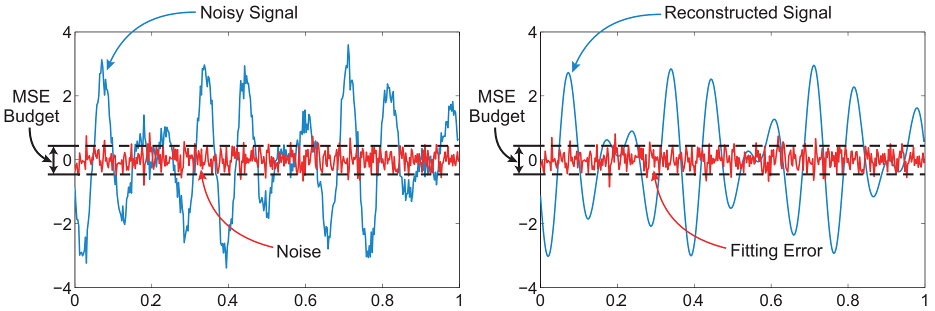
$$\looparrowright \quad \|\text{Data} - \text{Model}\|_2^2 \leq \text{MSE Budget}$$

$\looparrowright$ Treat all models that satisfy the budget as equal

Fitting using a MSE Budget

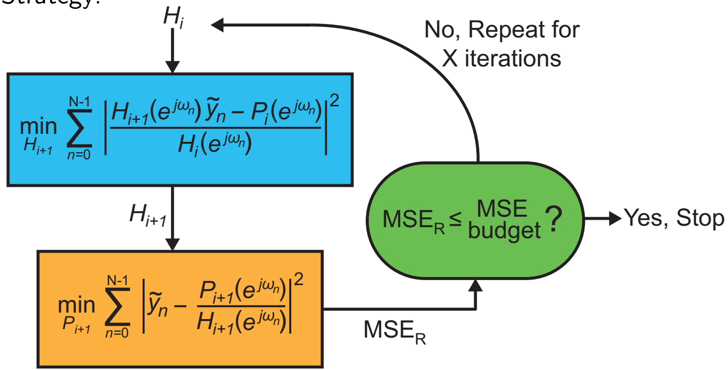


Obtaining the MSE Budget:

- ↬ Noise Level of Acquisition Device
- ↬ Estimated from Noisy Signal

Solving the Model-Fitting

Iterative Strategy:



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Solving the Model-Fitting

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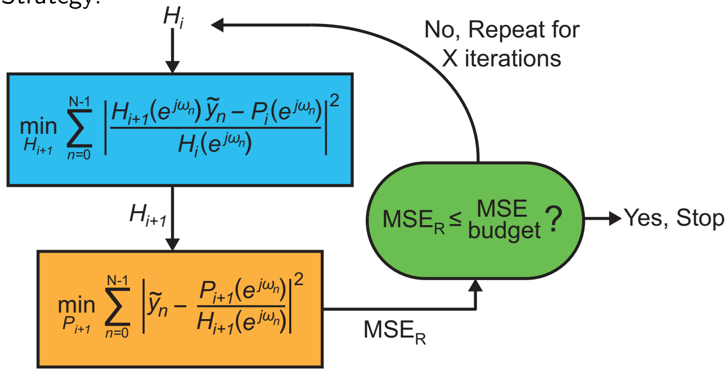

1. Update denominator H_{i+1} $\Longrightarrow$ Solve linear system of equations
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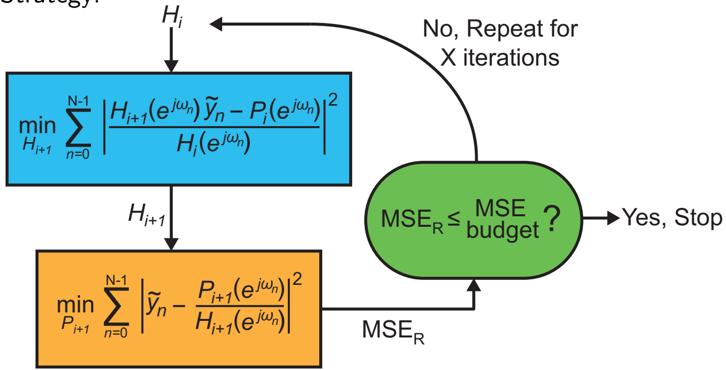
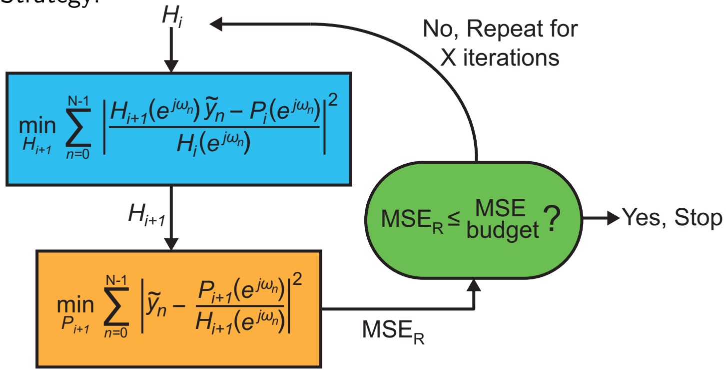


1. Update denominator $H_{i+1} \implies$ Solve linear system of equations
2. Update numerator $P_{i+1} \implies$ Obtain model update
3. Check model against MSE budget

Solving the Model-Fitting

Iterative Strategy:



Based on MSE budget $\Longrightarrow$ Not based on convergence

Solving the Model-Fitting

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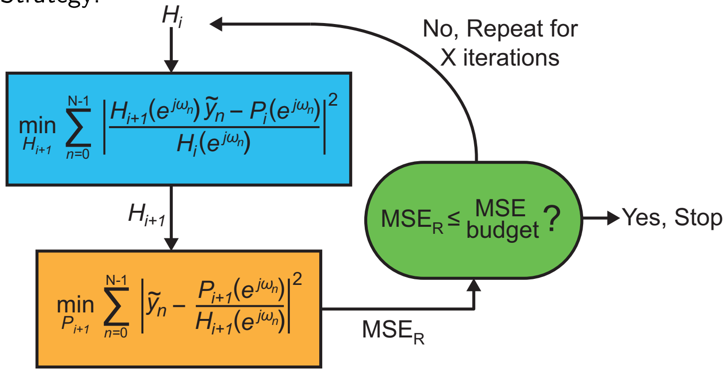

Reaching X iterations...

↬ Repeat with another random initialisation of H_0

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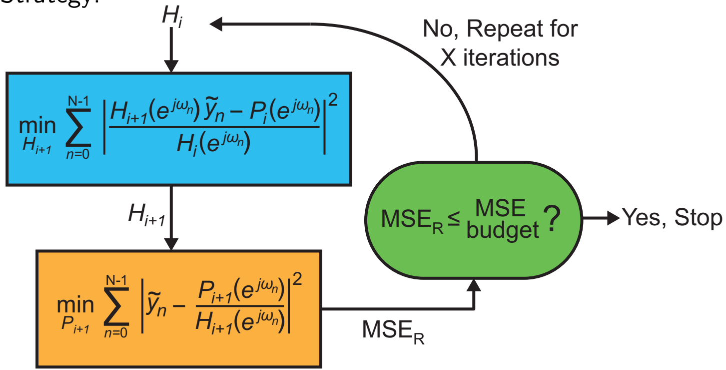

Reaching X iterations...

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Costly?....

Solving the Model-Fitting

No $\implies$ Very few initialisations are needed

Example of Robustness

FRI signal with $K = 6$ Diracs and $N = 51$ samples:

30 random initialisations $\implies$ Algorithm failure rate $= 0.015\%$
50 random initialisations $\implies$ Algorithm failure rate $= 0.0046\%$

$\looparrowright$ 15 random initialisations required to succeed in 99.9% cases
$\looparrowright$ 3 random initialisations required to succeed in 95% cases

*Results obtained using 500,000 realisations

Solving the Model-Fitting

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Finding the Rate of Innovation

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Fitting algorithm either succeeds or fails $\Longrightarrow$ Binary outcome

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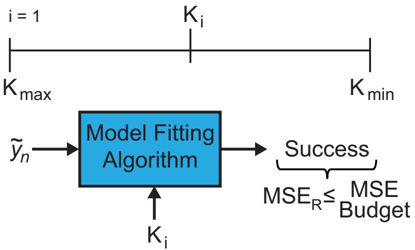
Dichotomic Algorithm:

Binary search to determine the minimum K, i.e. rate of innovation



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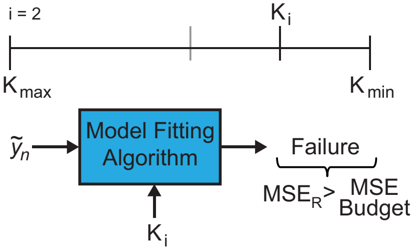

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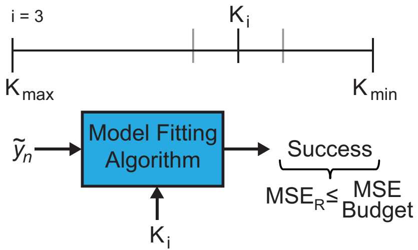

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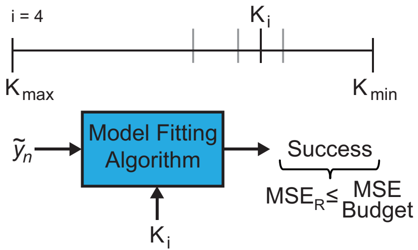

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$\looparrowright$ Possible to estimate a lower rate of innovation than the original

Finding the Rate of Innovation

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Concept of Parsimony:

The sparsest model is the most appropriate to represent the data

Simulation Results

Set-up:

- FRI signal with $K = 12$ Diracs and $N = 97$ samples
- Number of random initialisations = 50

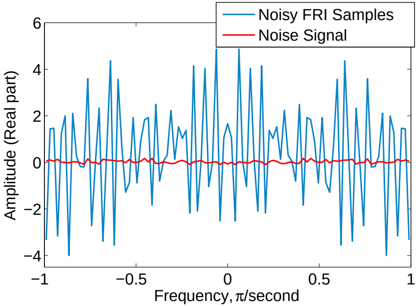
Compare against the Bayesian Information Criterion (BIC)[1]:

- Calculate the BIC for every K
- Choose K with the lowest BIC value

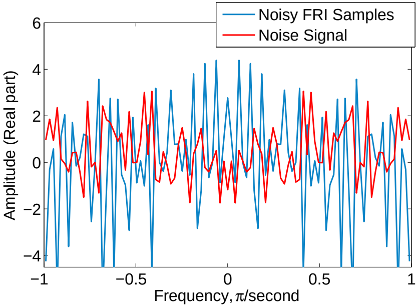
[1] P. Stoica & Y. Selen, '*Model-order selection*', IEEE Signal Processing Mag., 2004.

Simulation Results

Two sets of noisy conditions:



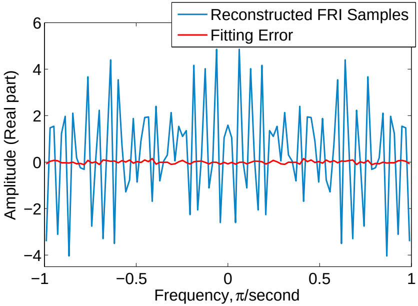
(a) Input SNR $= 30\text{dB}$



(b) Input SNR $= 5\text{dB}$

Simulation Results

Results of Dichotomic Algorithm:



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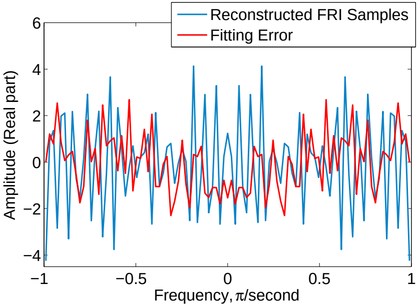

(b) Input SNR $= 5$dB

$$\text{Fit to MSE budget} \implies \underbrace{\text{Fitting Error}}_{\text{Noisy Data - Model}} \sim \text{Noise Signal}$$

Simulation Results

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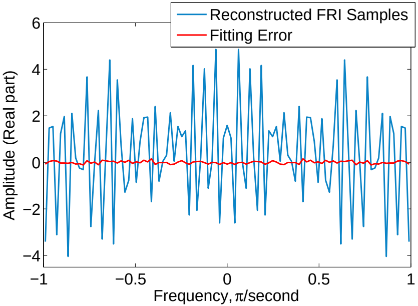
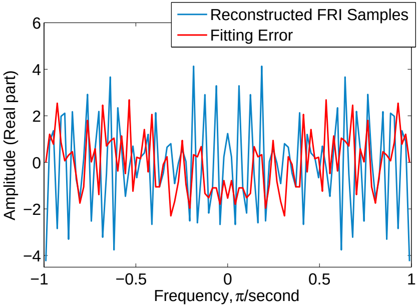


(a) Input SNR $= 30$dB

↬ Output SNR = 34.2dB

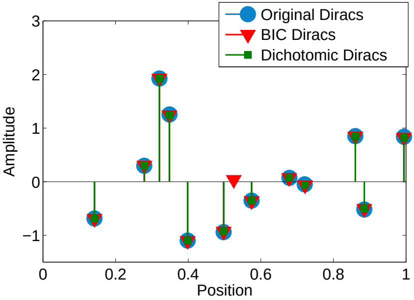


(b) Input SNR $= 5$dB

↬ Output SNR = 7.3dB

Simulation Results

Sparse parameters:



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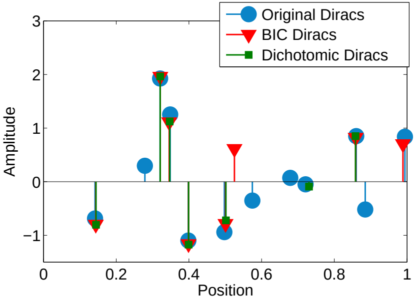


(b) Input SNR $= 5\text{dB}$

↬ BIC = 13 Diracs
↬ Dichotomic = 12 Diracs

↬ BIC = 8 Diracs
↬ Dichotomic = 7 Diracs

Conclusion

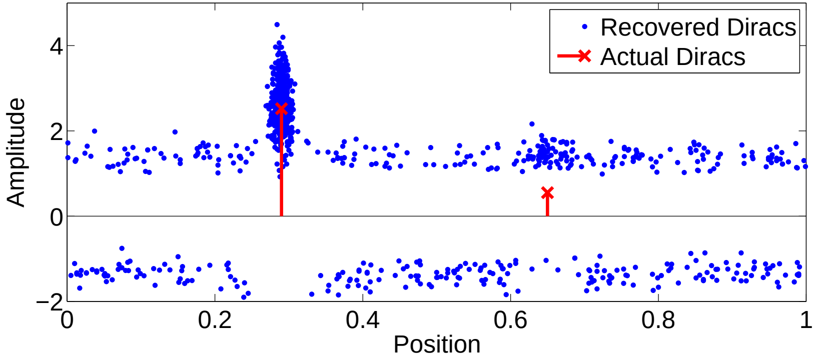
- Introduced the concept of FRI recovery based on a MSE budget
  - Model representation of the FRI samples $\Longrightarrow$ A ratio of polynomials
  - Fit model to noisy data until MSE budget is satisfied
  - Fitting algorithm is fast, accurate and robust
- Presented framework for finding rate of innovation
  - Outcome of model-fitting is binary $\Longrightarrow$ Succeeds or fails
  - Design binary search for the rate of innovation
  - Parsimony $\Longrightarrow$ Sparsest model is best
- Results demonstrating the advantages of the algorithm
  - Obtained FRI signals that met the MSE budget
  - Sparse parameters consistent with original

The End

Thank you for listening

Possibility of Losing Innovation

Recovering 2 Diracs in heavy noise, $\mathrm{SNR} = 0\mathrm{dB}$:



↬ Estimation using Maximum Likelihood over 500 realisations