Fitting Instead of Annihilation: Improved Recovery of Noisy FRI Signals

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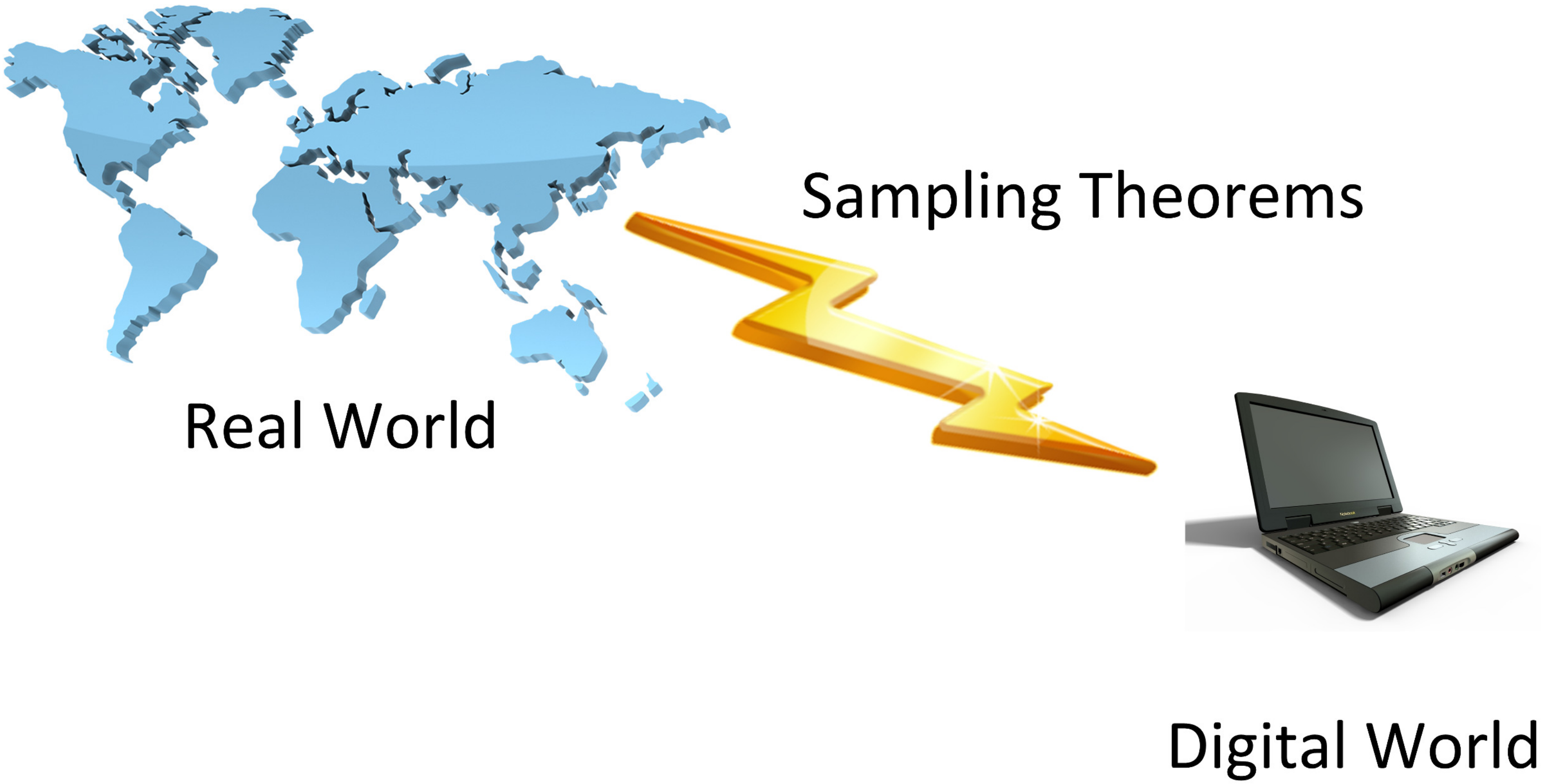
Overview

- Signal Acquisition $\Longrightarrow$ Real World to Digital World
- Signals with Finite Rate of Innovation (FRI)
  - Sampling Framework
  - Reconstruction Procedure
  - Model Mismatch
- Fitting Approach to FRI Reconstruction
  - MSE criteria
  - FRI Samples as a Ratio of Polynomials
  - Algorithm
- Simulation Results
- Conclusions

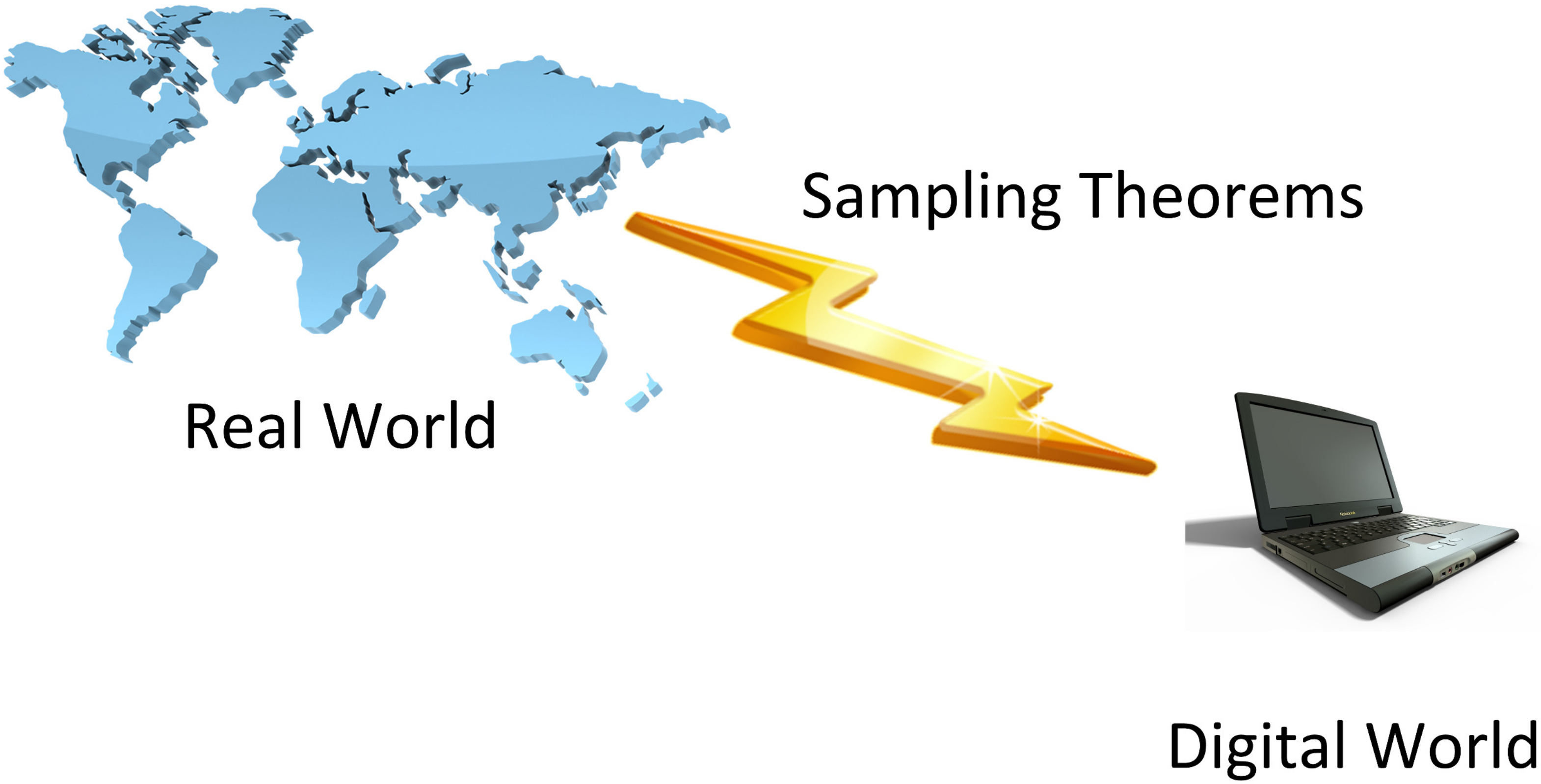
Acquiring Signals



Acquiring Signals $\Longrightarrow$ Transition from continuous to discrete domains

Sampling Theorems $\Longrightarrow$ Act as bridge between domains

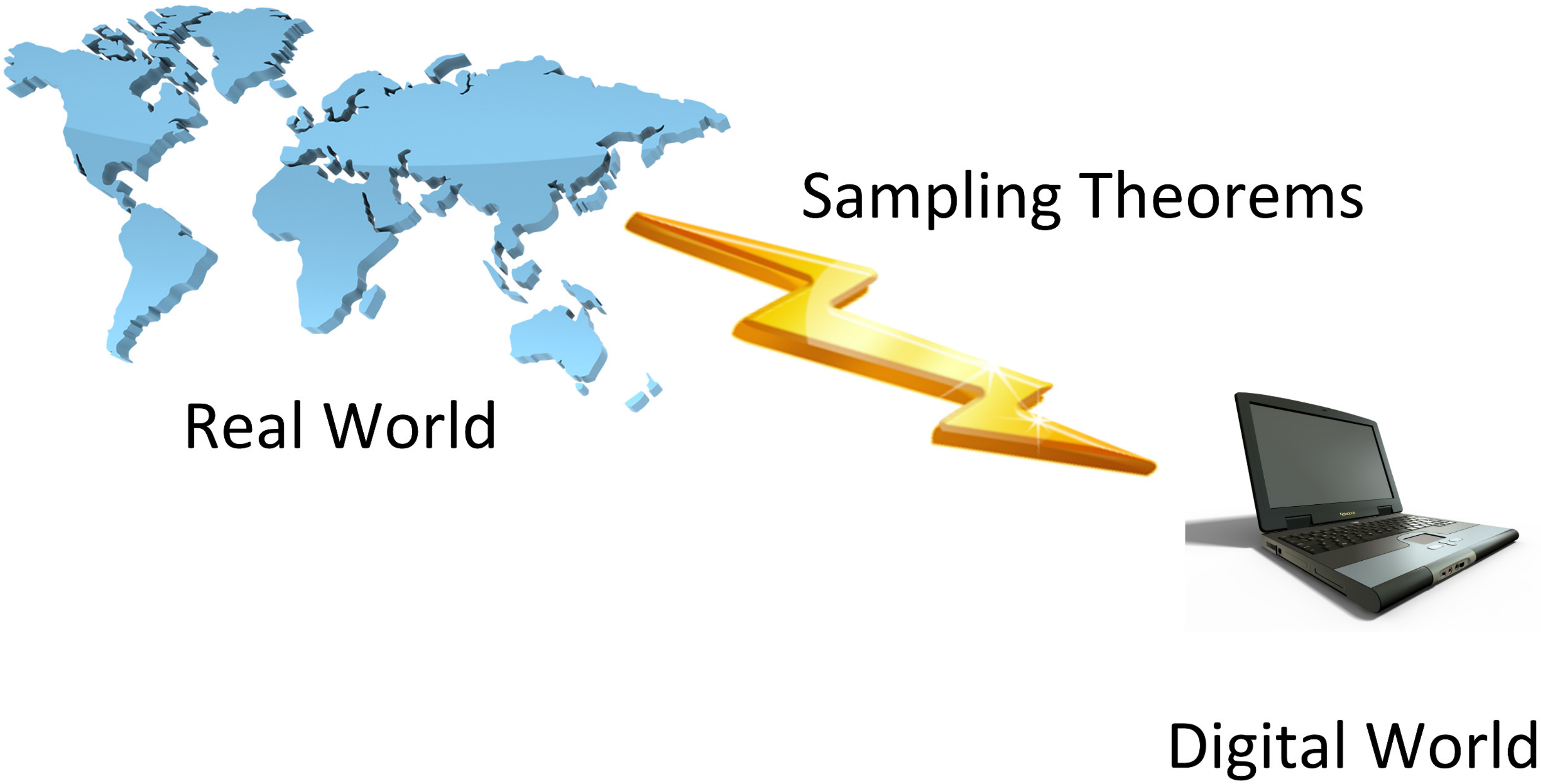
Acquiring Signals



Goal $\Longrightarrow$ Lossless transition between these domains

$\looparrowright$ Example: Sampling bandlimited signals

Acquiring Signals



Recently:

↬ Perfect reconstruction for a class of non-bandlimited signals

Finite Rate of Innovation

Signals that possess a finite number of degrees of freedom per unit time

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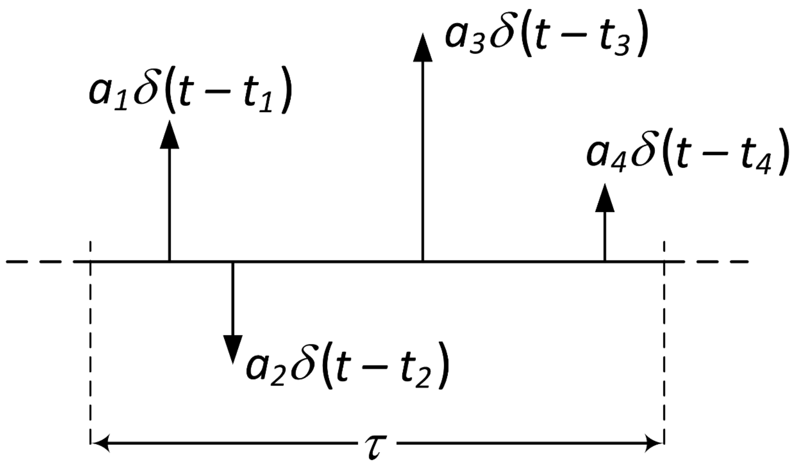
Signals that possess a finite number of degrees of freedom per unit time

For example:



Completely defined by 2 parameters $\implies$ An amplitude a_k and position t_k

Finite Rate of Innovation

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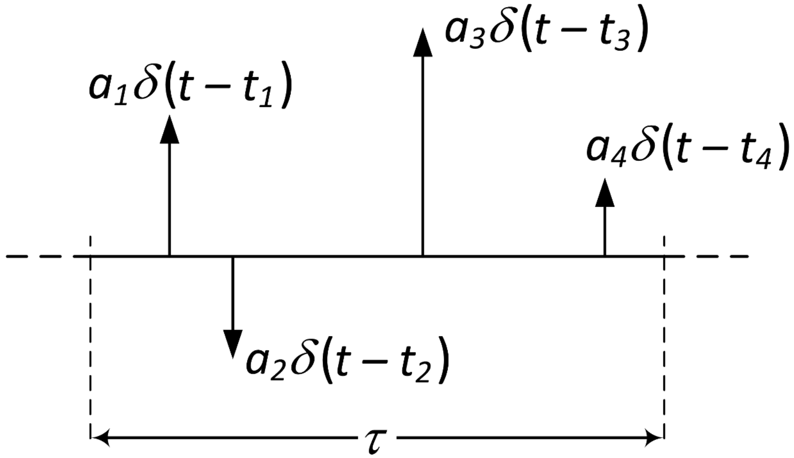
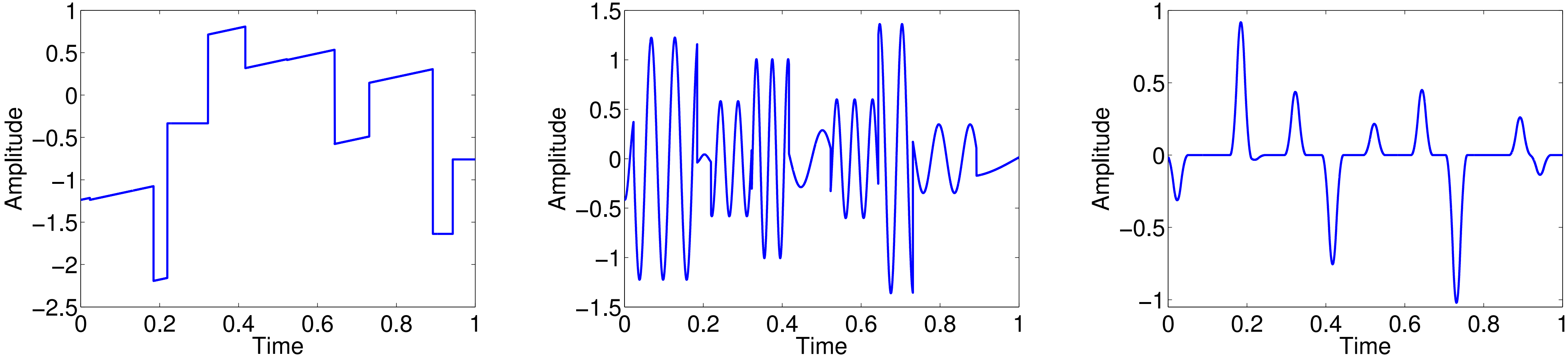


Completely defined by 2 parameters $\Longrightarrow$ An amplitude a_k and position t_k

$\looparrowright$ For K Diracs $\longrightarrow$ Rate of Innovation = $2K/\tau$

Beyond Diracs...

Other 1D FRI signals:



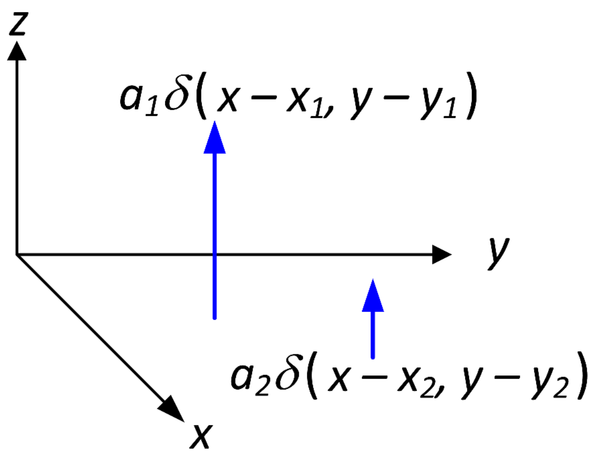
(a) Piecewise Polynomials (b) Piecewise Sinusoids (c) Stream of Pulses

Applications in Bio-medics:

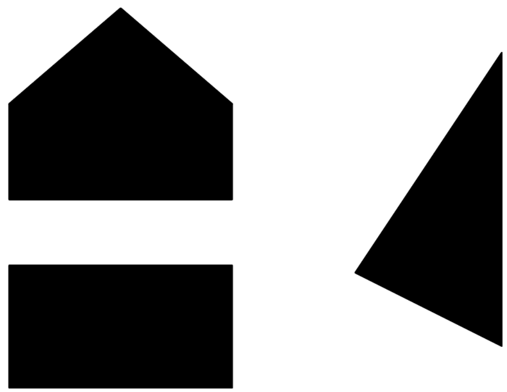
- Spike Estimation in Neurophysiological Data
- Compression of EEG and ECG signals

Beyond Diracs...

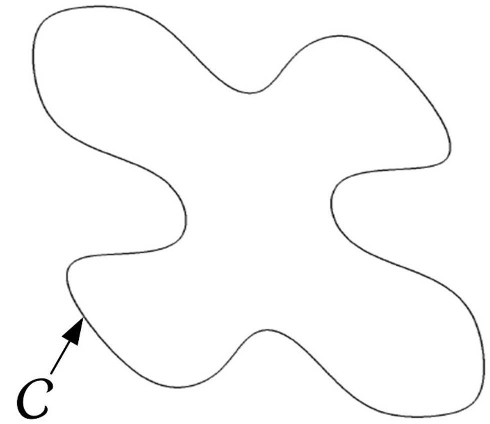
Higher Dimensional FRI signals:



(a) 2D Diracs



(b) Bilevel Polygons

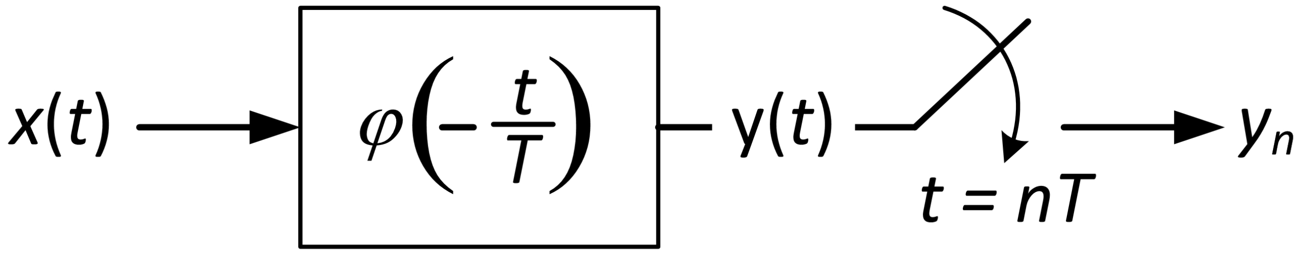


(c) Curves implicitly define by $\mathcal{C} : \mu(x, y) = 0$

Applications in image processing:

- Image Up-Sampling
- Image Super Resolution

FRI Sampling Framework



Periodic Stream of K Dirac:

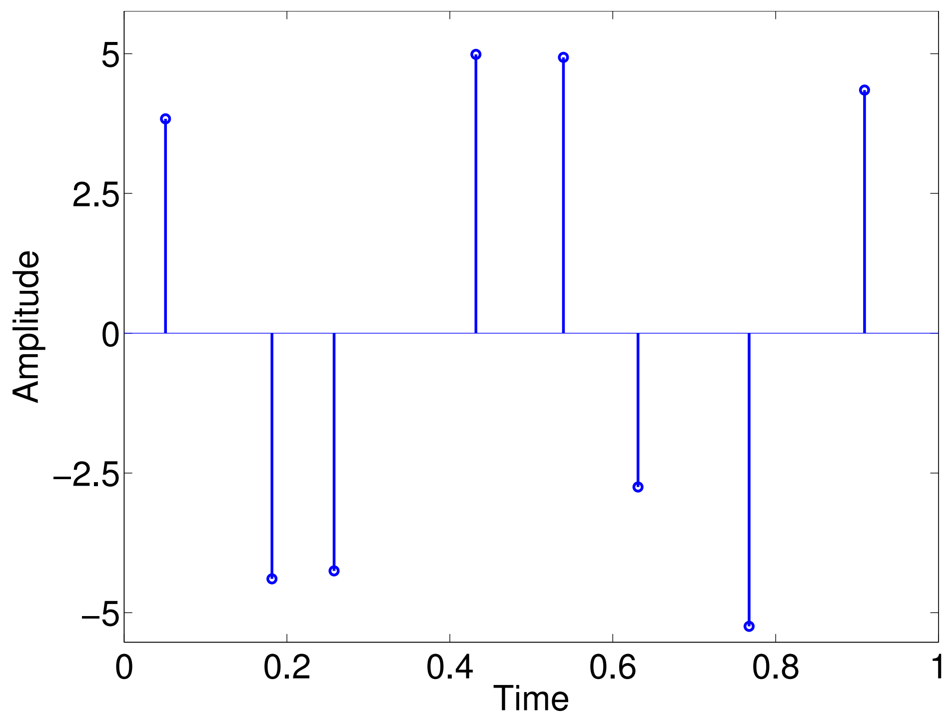
$$x(t) = \sum_{k=1}^{K} \sum_{l \in \mathbb{Z}} a_k \delta(t - t_k - l\tau),$$

↬ Rate of Innovation $= 2K/\tau$

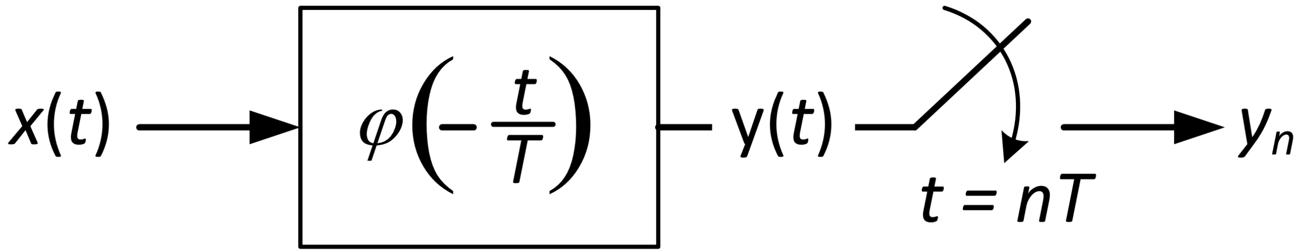
$a_k \longrightarrow$ Amplitudes

$t_k \longrightarrow$ Positions

$\tau \longrightarrow$ Period



FRI Sampling Framework



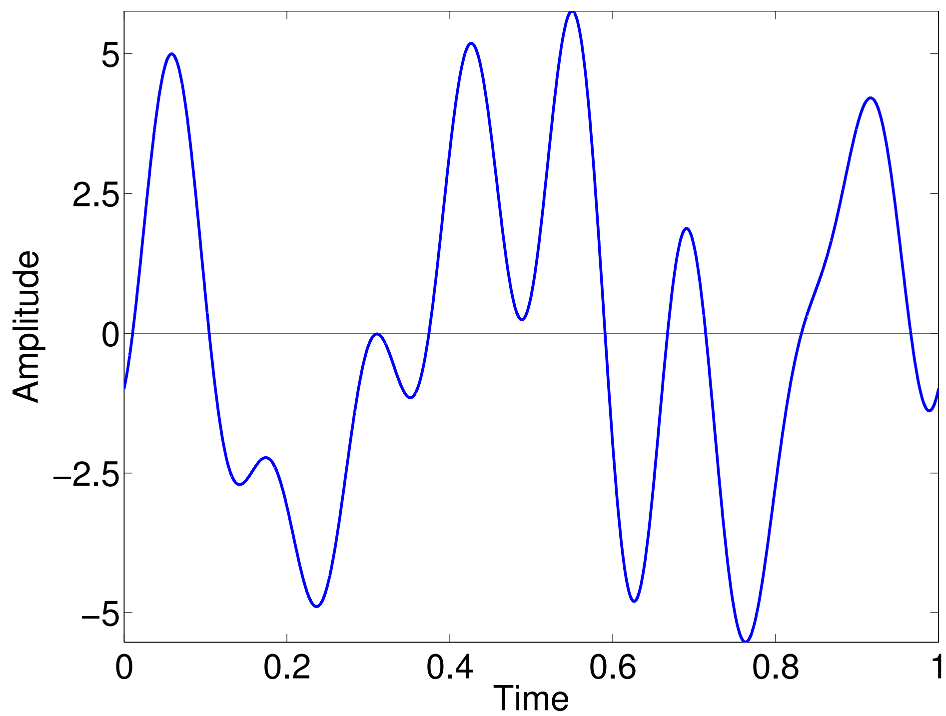
Filtered using Sampling Kernel:

$$\varphi\left(\frac{t}{T}\right) = \text{sinc}\left(\frac{\pi t}{T}\right) = \text{sinc}\left(\pi B t\right)$$

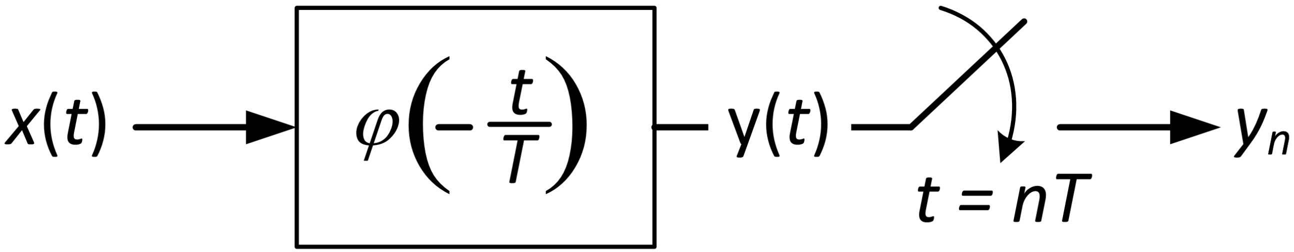
where,

$T \longrightarrow$ Sampling Period

$B \longrightarrow$ Bandwidth (an odd number)



FRI Sampling Framework

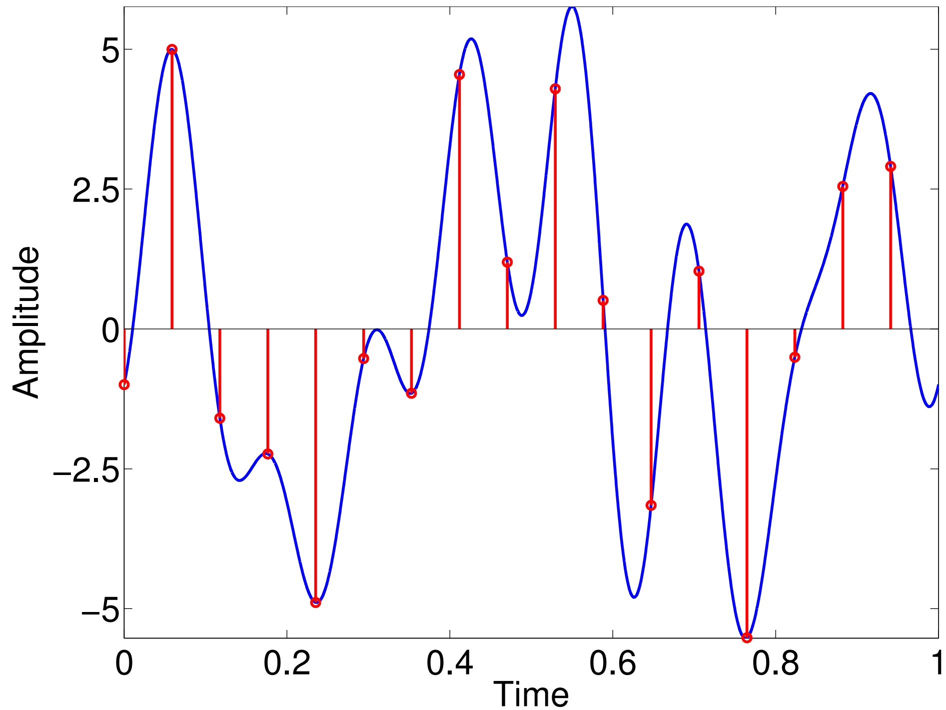


Discrete Samples:

$$y_n = \int_{-\infty}^{\infty} x(t)\text{sinc}\big(\pi B(nT - t)\big)\, \mathrm{d}t$$

Due to the periodic nature of $x(t)$ ...

$$y_n = \sum_{k=1}^{K} a_k \underbrace{\frac{\sin(\pi B(nT - t_k))}{B\sin(\pi(nT - t_k))}}_{\text{Dirichlet Kernel}}$$



Reconstruction Procedure

The Fourier Transform of the Samples:

$$\hat{y}_m = \sum_{n=0}^{N-1} y_n e^{-j2\pi mn/N} = \begin{cases} \sum_{k=1}^{K} a_k e^{-j2\pi m t_k} & \text{if } |m| \leq \lfloor B/2 \rfloor \\ 0 & \text{for other } m \in [-N/2, N/2] , \end{cases}$$

Note that $\tau = 1$.

Reconstruction Procedure

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Two step reconstruction:

1) Non-linear Recovery of $t_k \implies$ Annihilation Method

Determine filter $H(z) = \sum_{k=0}^{K} h_k z^{-k}$ such that $\underbrace{h_m * \hat{y}_m = 0}_{\text{Annihilation}}$

$\looparrowright$ Roots of filter H define Dirac positions

2) Linear Recovery of a_k

Using $t_k \implies$ Determine a_k via least mean squares

Reconstruction Procedure

Annihilation Method in more detail:

$$h_m * \hat{y}_m = 0 \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} \hat{y}_K & \hat{y}_{K-1} & \cdots & \hat{y}_0 \\ \hat{y}_{K+1} & \hat{y}_K & \cdots & \hat{y}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{N-1} & \hat{y}_{N-2} & \cdots & \hat{y}_{N-K-1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{K+1} \end{bmatrix}}_{\mathbf{h}} = 0$$

Solve Annihilation Equation: $\mathbf{Ah} = 0 \quad \longrightarrow$ Requires $N = 2K$ samples

Reconstruction Procedure

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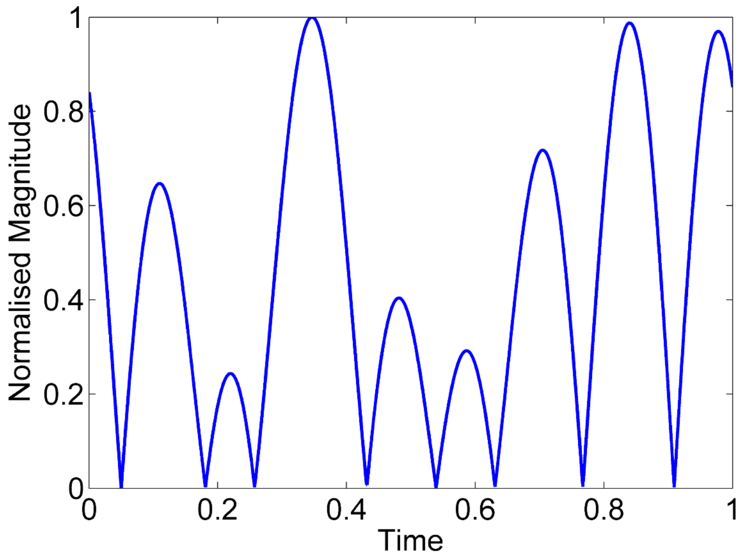
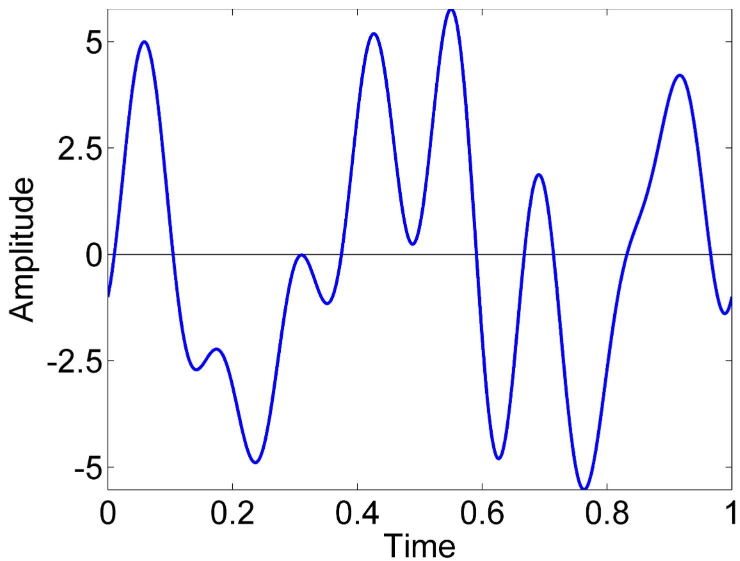


(a) Annihilation Filter

$\times$



(b) FRI Samples

Model Mismatch

An imperfect world:

Sample Corruption: $\tilde{y}_n = y_n + \epsilon_n$ therefore $\tilde{\mathbf{A}}\mathbf{h} \neq 0$

$\looparrowright$ Need a robust way of estimating the annihilation filter

Model Mismatch

An imperfect world:

Sample Corruption: $\tilde{y}_n = y_n + \epsilon_n$ therefore $\tilde{\mathbf{A}}\mathbf{h} \neq 0$

Two approaches:

i) Iterative Techniques $\Longrightarrow$ Cadzow's method[1]

- Observation: Matrix $\mathbf{A}$ is rank-K and Toeplitz when no noise
- Enforce rank-K and Toeplitz structure
- Constraint $\|\mathbf{h}\|^2 = 1$

ii) Non-iterative techniques $\Longrightarrow$ Matrix Pencil Method[2]

- Subspace method that directly estimates t_k
- Similarity to ESPRIT

[1] T. Blu, et al, '*Sparse sampling of signal innovations*', IEEE Signal Processing Mag, 2008
[2] I. Maravic and M. Vetterli, '*Sampling and reconstruction of signals with finite rate of innovation in the presence of noise*', IEEE Trans. Signal Processing, 2005

Model Mismatch

An imperfect world:

Sample Corruption: $\tilde{y}_n = y_n + \epsilon_n$ therefore $\tilde{\mathbf{A}}\mathbf{h} \neq 0$

Issues:

- Both approaches require large SVD
- Size of SVD scales with the number of samples N
- Difficult to handle large number of samples

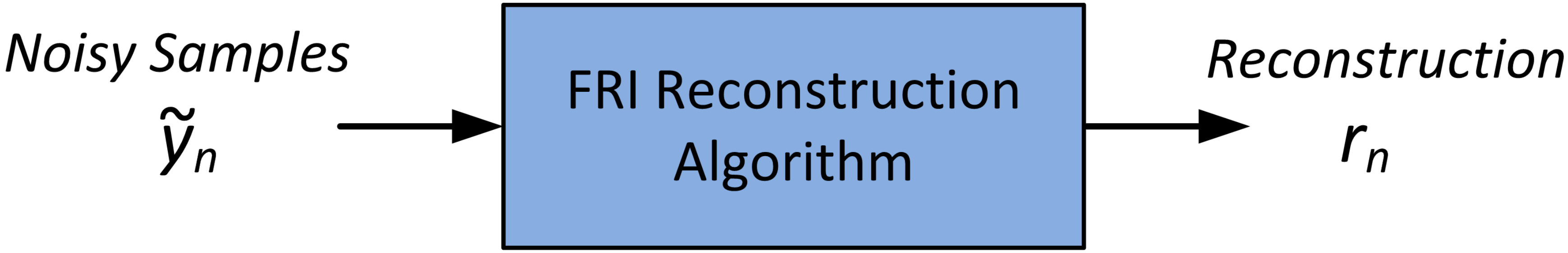
Proposed Approach:

$\looparrowright$ Fast algorithm

$\looparrowright$ Scales with K

$\looparrowright$ Reliable at low SNR

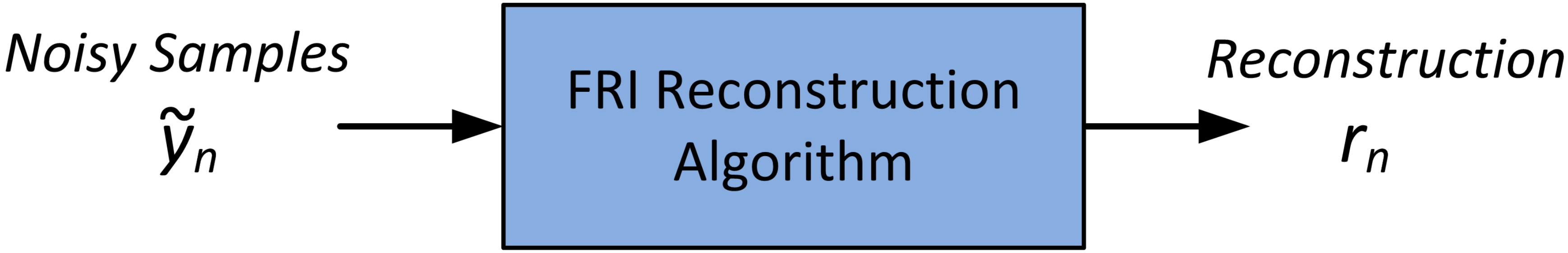
Fitting using the Mean Squared Error



Known quantity:

Mean Squared Error (MSE) between the original and noisy samples $\Longrightarrow$ MSE $(y_n, \tilde{y}_n)$

Fitting using the Mean Squared Error



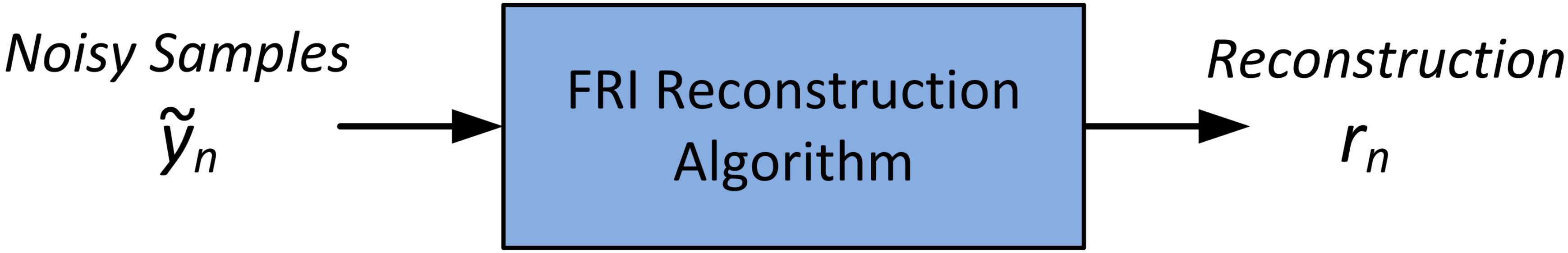
Known quantity:

Mean Squared Error (MSE) between the original and noisy samples $\Longrightarrow$ MSE $(y_n, \tilde{y}_n)$

Able to calculate:

MSE between the noisy and reconstructed samples $\Longrightarrow$ MSE $(\tilde{y}_n, r_n)$

Fitting using the Mean Squared Error



Proposed framework:

Determine r_n

such that

↬ $\boxed{\text{MSE}\left(\tilde{y}_n, r_n\right) \leq \text{MSE}\left(y_n, \tilde{y}_n\right)}$

↬ r_n is a FRI signal

Polynomial Representation of FRI Samples

Noise-free FRI Samples:

$$y_n = \sum_{k=1}^{K} a_k \underbrace{\frac{\sin(\pi B(nT - t_k))}{B\sin(\pi(nT - t_k))}}_{\text{Dirichlet Kernel}}$$

Polynomial Representation of FRI Samples

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Using Euler's formulas $\Longrightarrow$ FRI samples represented as the ratio of two polynomials

$$y_n = z^{nM} \frac{P(z^n)}{H(z^n)} = z^{nM} \frac{\sum_{l=0}^{K-1} p_l z^n}{\sum_{k=0}^{K} h_k z^n}$$

where

- P is a polynomial of order $K - 1$
- H is the annihilation filter, a polynomial of order K
- $z^n = e^{j2\pi n/N}$
- $M = N/2$

Polynomial Representation of FRI Samples

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Notice:

$\looparrowright$ FRI signal defined by the coefficients of the polynomials

$\looparrowright$ Order of Polynomials dependent on number of Diracs K

$\looparrowright$ Representation is in the time domain

The Fitting Algorithm

Combining MSE criteria and polynomial representation:

$$\min_{H,P} \sum_{n=0}^{N-1} \left| \tilde{v}_n - \frac{P\left(e^{j2\pi n/N}\right)}{H\left(e^{j2\pi n/N}\right)} \right|^2$$

where H and P are defined by coefficients $\mathbf{h}$ and $\mathbf{p}$, and $\tilde{v}_n = \tilde{y}_n e^{-j2\pi nM/N}$

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where H and P are defined by coefficients $\mathbf{h}$ and $\mathbf{p}$, and $\tilde{v}_n = \tilde{y}_n e^{-j2\pi nM/N}$

Non-linear minimisation $\implies$ Choose to solve linearly in an iterative manner

$$\min_{\mathbf{h}_i,\mathbf{p}} \sum_{n=0}^{N-1} \left| \frac{H_i\left(e^{j2\pi n/N}\right)\tilde{v}_n - P\left(e^{j2\pi n/N}\right)}{H_{i-1}\left(e^{j2\pi n/N}\right)} \right|^2$$

where i is the iteration number

Note similar to the Steiglitz-McBride algorithm and Sanathanan-Koerner algorithm

The Fitting Algorithm

Step 1) Initialisation: Calculate $\mathbf{h}_0 \implies$ Total least squares solution or set $h(0) = 1$

Step 2) Solve for $\mathbf{h}_i$:

$$\min_{\mathbf{h}_i} \sum_{n=0}^{N-1} \left| \frac{H_i\left(e^{j2\pi n/N}\right)\tilde{v}_n - P\left(e^{j2\pi n/N}\right)}{H_{i-1}\left(e^{j2\pi n/N}\right)} \right|^2$$

under constraint on $\mathbf{h}_i \implies \|\mathbf{h}\|^2 = 1$ or $h(0) = 1$

Step 3) Solve for $\mathbf{p}$:

$$\min_{\mathbf{p}} \sum_{n=0}^{N-1} \left| \tilde{v}_n - \frac{P\left(e^{j2\pi n/N}\right)}{H_i\left(e^{j2\pi n/N}\right)} \right|^2$$

Step 4) Assess MSE criteria $\implies$ Stop if MSE is below known noise variance

The Fitting Algorithm

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under constraint on $\mathbf{h}_i \implies \|\mathbf{h}\|^2 = 1$ or $h(0) = 1$

For $\|\mathbf{h}\|^2 = 1$:

↬ SVD of size $K + 1$

For $h(0) = 1$:

↬ Linear set of $K + 1$ equations

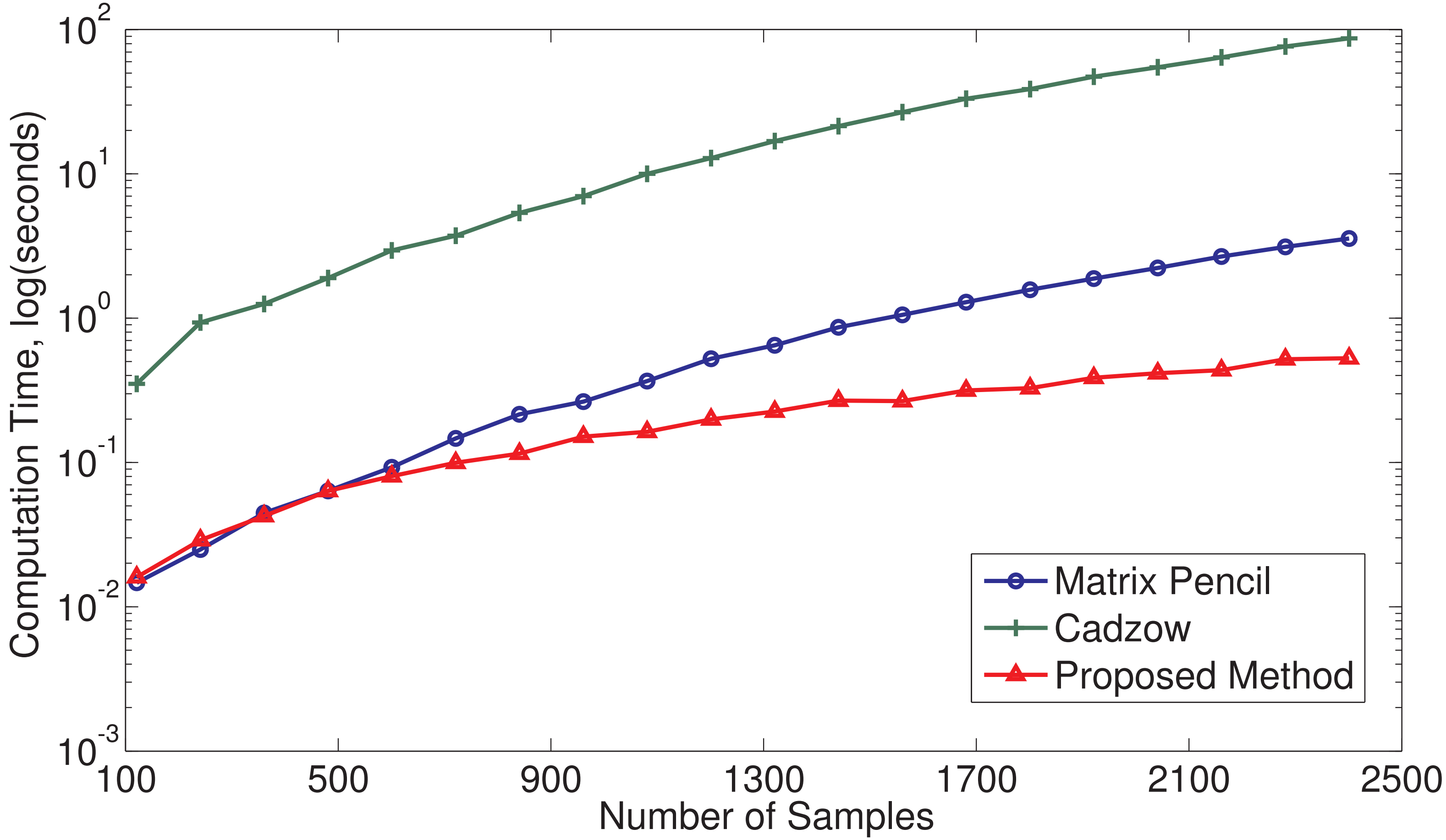
Step 3) Solve for $\mathbf{p}$:

$$\min_{\mathbf{p}} \sum_{n=0}^{N-1} \left| \tilde{v}_n - \frac{P\left(e^{j2\pi n/N}\right)}{H_i\left(e^{j2\pi n/N}\right)} \right|^2$$

Step 4) Assess MSE criteria $\implies$ Stop if MSE is below known noise variance

Simulation Results

Assessing computation time:

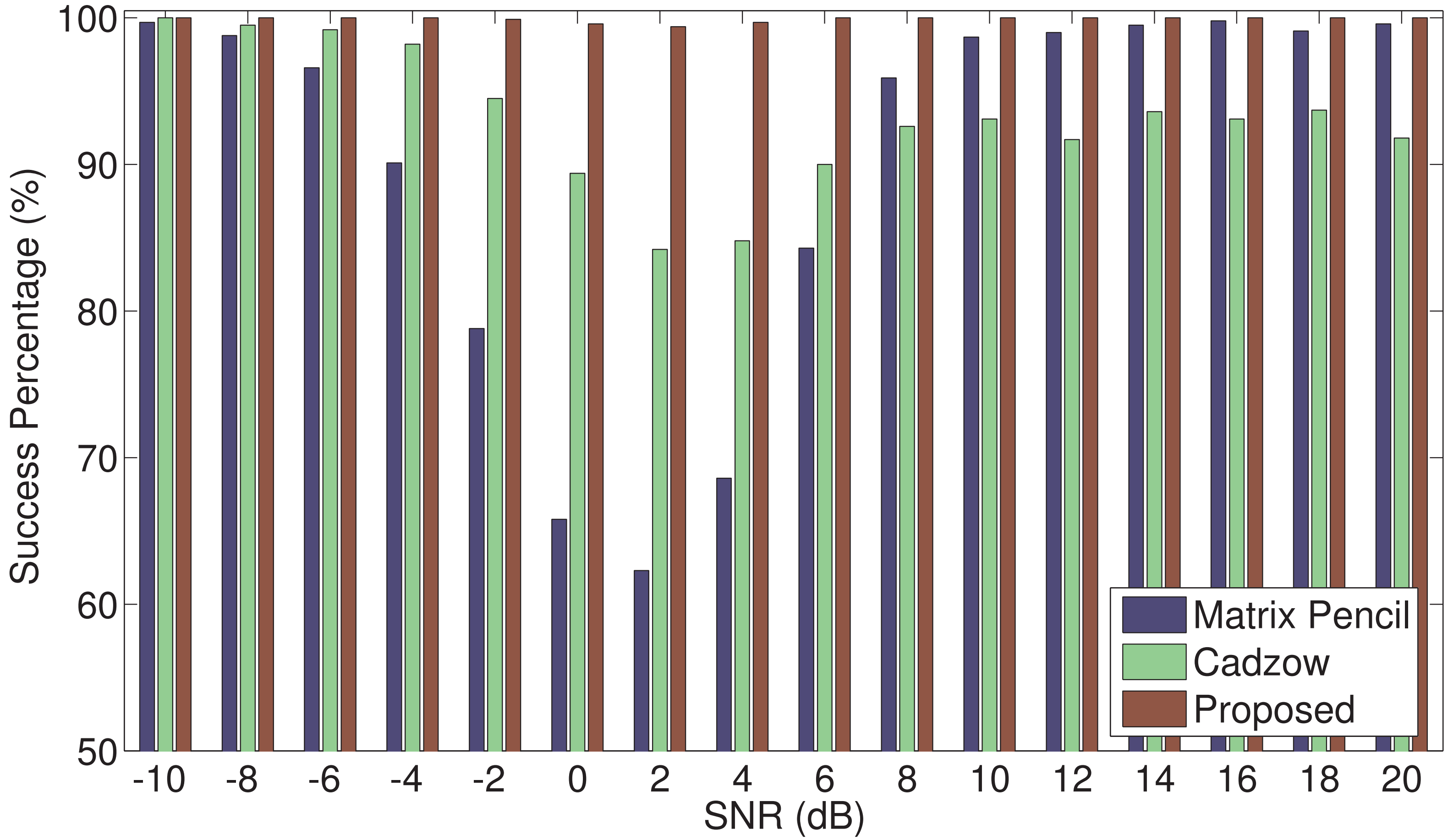


Structure of Signal: K = 60 Diracs

Noise Characteristics: SNR = 5dB

Simulation Results

Assessing the MSE criteria: Success = MSE criteria achieved

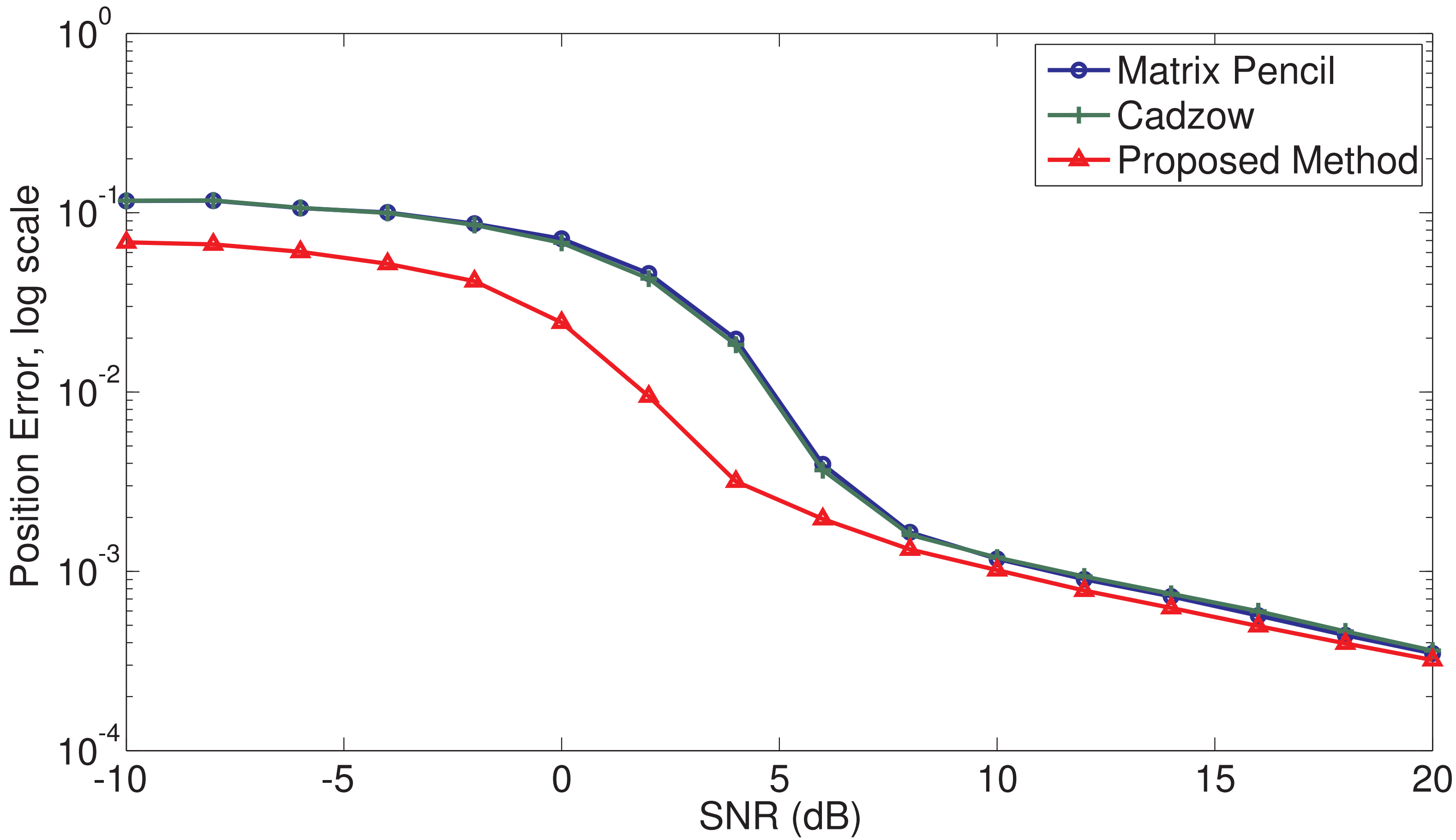


Structure of Signal: K = 6 Diracs, N = 51 samples

Noise Characteristics: 1000 realisations of noise each SNR

Simulation Results

How this relates to estimating the positions of the Diracs:



Structure of Signal: K = 6 Diracs, N = 51 samples

Noise Characteristics: 1000 realisations of noise each SNR

Conclusions

- Presented a new framework for reconstructing noisy FRI signals
  - MSE between reconstruction and noisy samples less than Input MSE
- Demonstrated that FRI samples can be represented as ratio of polynomial
  - Samples belong to a period stream of K Diracs
  - Order of Polynomials depend on K
- Presented new algorithm for reconstructing noisy FRI signals
  - Fit polynomial representation to the noisy FRI samples
  - Iterative algorithm
  - Based on MSE criteria
- Empirical results showing advantages over existing algorithms
  - Reduced computation time
  - Reliable results $\Longrightarrow$ MSE criteria
  - Good position estimation

The End

Thank you for listening